CS 261 Review Questions with answers

1. Let p, q, and r be the following statements:
   p = "Grizzly bears have been seen in the area."
   q = "Hiking is safe on the trail."
   r = "Berries are ripe along the trail."

   Write the following statement using logic symbols:
   "Hiking is not safe on the trail whenever grizzly bears have been seen in the area and berries are ripe along the trail."

   $(p \wedge r) \rightarrow \sim q$

2. Write the truth table for the following statement form: $(p \rightarrow q) \rightarrow r$.

p	*q*	*R*	*$p \rightarrow q$*	*$(p \rightarrow q) \rightarrow r$*
T	T	T	T	T
T	T	F	T	F
T	F	T	F	T
T	F	F	F	T
F	T	T	T	T
F	T	F	T	F
F	F	T	T	T
F	F	F	T	F

3. If the following statement form is true, then what can we conclude about the values of p, q and r?

   $$p \wedge (p \rightarrow \sim q) \wedge (q \vee r)$$

   This is an AND statement. An AND statement is only true whenever each factor being ANDED together is true. Therefore, we can conclude that p is true, $p \rightarrow \sim q$ is true, and that $q \vee r$ is true.

   Since $p \rightarrow \sim q$ is true and p is true, we know (from modus ponens) that $\sim q$ is true. In other words, q is false.

   Finally, since $q \vee r$ is true and q is false, we see that r has to be true.

   Therefore, p = true, q = false, and r = true.

4. True or false: If a propositional statement is not a contradiction, then it is a tautology. Justify your answer.

   False. It's possible for a statement to be neither a tautology nor a contradiction. This happens for a statement that is true for some assignment of truth values to the variables, and false for others.

5. Write the contrapositive of this statement: "If x is noble and fast, then x is a horse."

If x is not a horse, then x is not noble or x is not fast.

6. Consider the following statement: "If you are making a reservation, then you must pay a deposit."
   a. Rewrite the statement so that it contains the phrase "necessary condition"

      A necessary condition for making a reservation is that you pay a deposit.

   b. Write the converse, inverse and contrapositive of the statement.

      Converse: If you must pay a deposit, then you are making a reservation.
      Inverse: If you are not making a reservation, then you do not have to pay a deposit.
      Contrapositive: If you don't have to pay a deposit, then you are not making a reservation.

   c. Rewrite the original statement without using "if…then". (i.e. convert to "and" or "or" statement, as appropriate.)

      You are not making a reservation, or you must pay a deposit.

7. Consider this statement: "If Martha has a loaf of bread, then she also has peanut butter and jelly."
   a. Re-write the statement so that it uses the phrase "sufficient condition."

      Martha having a loaf of bread is a sufficient condition for her also to have peanut butter and jelly.

   b. Write the negation of the original statement.

      Martha has a loaf of bread, but she does not have either peanut butter or jelly.

8. Negate these statements:
   a. This is important tax information and is being furnished to the Internal Revenue Service.

      This is not important tax information or it is not being furnished to the Internal Revenue Service.

   b. If you are required to file a return, a negligence penalty or other sanction may be imposed on you if this income is taxable and the IRS determines it has not been reported.

      Let's first write this given statement in symbols:
      Required → ((Taxable ^ ~Reported) → (Penalty or Sanction))

      Negation:
      Required ^ ~((Taxable ^ ~Reported) → (Penalty or Sanction))
      = Required ^ (Taxable ^ ~Reported) ^ ~(Penalty or Sanction)
      = Required ^ Taxable ^ ~ Reported ^ ~Penalty ^ ~Sanction

Negation in English:
You are required to file a return, and this income is taxable, and the IRS determines that it has not been reported, and neither a negligence penalty nor other sanction was imposed on you.

9. Consider this statement P = “if today is Saturday and it is raining, then we will watch a movie.”
   a. Write the contrapositive of P.

      If we do not watch a movie, then today is not Saturday or it is not raining.

   b. Write the negation of P.

      Today is Saturday and it is raining, but we will not watch a movie.

   c. Write the converse of P.

      If we watch a movie, then today is Saturday and it is raining.

10. Consider this statement:
    P = “If Bob rolls the dice, then he can either move his token forward or accept a challenge.”
    a. Write the inverse of P.

       If Bob does not roll the dice, then he cannot move his token forward and cannot accept a challenge.

    b. Write the contrapositive of P.

       If Bob cannot move his token forward and cannot accept a challenge, then he did not roll the dice.

    c. Write the converse of P.

       If Bob can move his token forward or accept a challenge, then he rolled the dice.

    d. Write the negation of P.

       Bob rolls the dice, and he cannot move his token forward, and cannot accept a challenge.

11. Determine if each of the following statement forms is a tautology.
    a. (((p or q) → r) and (~p)) → (q → r)

       Let’s see if there is any way to make this statement false.
       An implication is false when it’s of the form T → F.

Therefore: ((p or q) → r) and (~p)) is true, and (q → r) is false.
Since q → r is false, we immediately see that <u>q = true</u> and <u>r is false</u>.

An "and" statement is true when both parts are true. Therefore:
((p or q) → r) is true, and (~p) is also true.
Since ~p is true, <u>p is false</u>.

Next, we see that ((p or q) → r) is true, yet r is false. If an implication evaluates to true but the conclusion is false, then it must be of the form F → F. Therefore (p or q) must be false. An "or" statement is false when both parts are false. This means that both p and q are false. But this contradicts our earlier observation that q = true.

Thus, there is no way for the statement form to be false. We conclude that it must be a tautology.

b. (A → (B → C)) → ((A → B) → (A → C))

Suppose it is not a tautology. That means there would be some way to assign truth values A, B and C to make the statement form false. If the whole statement is false, then it must be of the form T → F, and so the second half of the statement, ((A → B) → (A → C)), is false.

If (A → B) → (A → C) is false, then A → B is true and A → C is false. Since A → C is false, right away we see that A = true and C = false. Next, since A → B is true and A is true, the B is also true.

Now let's look at the first half of the statement, using what we know about A, B and C. The first half, A → (B → C) must be true in order to make the entire statement false. Let's substitute: true → (true → false). This evaluates to false, and so we have a contradiction.

Thus, there is no way for the statement form to be false, so it must be a tautology.

c. (q ^ r) v ((p → q) ^ ~r) v ~q

Let's see if this statement form could ever be false. This statement is three terms separated by OR operators. The only way the entire statement could be false is for each term to be false. In particular, ~q would be false, which means q is true. The first term, (q ^ r) is false but q is true. This suggests that r must be false. The second term is a conjunction that must be false, and ~r is true, so p → q must be false. We see p → q is false and we know q is true. But this is impossible, so we have arrived at a contradiction. Therefore, the statement form is a tautology.

d. (p → ~q) ∨ (q → ~r) ∨ (r → p)

Is it ever false? Suppose it's false. It is three statements ORed together. OR only returns false if the operands are false. Therefore:

*p → ~q is false. This means **p = true**, q = true.*
q → ~r is false. This means q = true, r = true.
*r → p is false. This means r = true, **p = false**.*

Notice the contradiction regarding the value of p. We conclude that the given statement form cannot be false. So, it is a tautology.

12. Give an example of an argument that uses *modus ponens*. Is your argument valid?

Modus ponens is a valid argument form. In other words, any argument written in this structure is guaranteed to be valid. An example is:

If it is raining, then there are clouds.
It is raining.
--
There are clouds.

13. Determine if the following arguments are valid or invalid. Explain your reasoning.

a. (p → q) and (r → s)
~q or ~s

~p or ~r

Let's create a statement form out of this argument, and see if it's a tautology. We obtain: (((p → q) ^ (r → s)) ^ (~q or ~s)) → (~p or ~r). Can this statement form ever be false?

An implication is false when it is of the form T → F. This means that the conclusion, (~p or ~r), must be false. An "or" statement is false when both pieces are false. Therefore, ~p and ~r are both false, so that p is true and r is true.

Next, we look at the left side of the big implication. It's an "and" statement, and it must be true. This can only happen when both pieces are true. Thus, the first part of the statement is true: ((p → q) ^ (r → s)). And we can break it down again, seeing that we have another "and" statement: p → q is true and r → s is true.

Now, let's use what we learned earlier about p and r. Since p → q is true and p is true, by modus ponens, q is true. Similarly, since r → s is true and r is true, s is true. We have now determined truth values for all 4 variables. Let's make sure they work for the part of the expression we have not yet evaluated.

Consider (~q or ~s). This must be true in order to make the entire statement form false. We already know that all 4 variables are true. Therefore ~q or ~s is false. We have arrived at a contradiction: the statement cannot be both true and false. Thus, the big statement form is a tautology, and so the argument is valid.

b. If the program compiles, then I can ski.
If I cannot find my goggles, then I cannot ski.
Therefore, if I cannot find my goggles, then the program does not compile.

Let's write this argument in symbols as follows.

c → s
~g → ~s

~g → ~c

Since the 2nd and 3rd statements have a lot of negations, let's rewrite them using the contrapositive to simplify them:

c → s
s → g
----
c → g

This argument is valid because it is the form of a common valid form, called transitivity.

c. "There are three big stores in the mall: Eaton's, Holt Renfrew, and The Bay. Eaton's is such a rotten place. And I can't afford Holt Renfrew. So, we'll just go to The Bay."

In symbols, this argument becomes:

e or h or b
~e
~h
----
b

With two applications of disjunctive syllogism, we see that this is a valid argument. First we observe that these two premises

e or (h or b)
~e

allow us to conclude that (h or b) is true. Next, our argument becomes:

h or b
~h
----
b

And we see that this is a valid argument form.

d. If you send me an e-mail message, then I will finish writing the program.
If you do not send me an e-mail message, then I will go to sleep early.
If I go to sleep early, then I will wake up feeling refreshed.
Therefore, if I do not finish writing the program, then I will wake up feeling refreshed.

The argument has this form:
Send → Finish
~Send → Sleep
Sleep → Refreshed

~Finish → Refreshed

If we take the contrapositive of the first statement, we obtain: *~Finish → ~Send*
~Send → Sleep
Sleep → Refreshed

~Finished → Refreshed

The argument is valid, because after making this change, we see that it is just two applications of the transitivity argument form.

e. If the robot can bake a cake, then I can leave the ship.
If aliens are approaching, then I cannot leave the ship.
Therefore, if aliens are not approaching, then the robot can bake a cake or I can leave the ship.

This argument takes the form
$p \to q$
$r \to \sim q$
Therefore, $\sim r \to (p \vee q)$

The argument is valid if and only if the following statement form is a tautology:
$((p \to q) \wedge (r \to \sim q)) \to (\sim r \to (p \vee q))$
Suppose it is false. Then, $(p \to q) \wedge (r \to \sim q)$ *is true, so that* $p \to q$ *is true and* $r \to \sim q$ *is true. Also, we have* $\sim r \to (p \vee q)$ *is false. This means* $\sim r$ *is true and* $p \vee q$ *is false. Thus,* p = *false,* q = *false and* r = *false. There is no contradiction with the other implications. We found a case that makes the statement false, so the corresponding argument is not valid.*

f. Nigel visited at least one of these three cities: Leeds, Manchester, Southampton.
If Nigel visited Leeds, then he saw a cricket match.
If Nigel visited both Southampton and Manchester, then he did not see a cricket match.
Therefore, if Nigel saw a cricket match, then Nigel visited Leeds.

The argument has this form:
$L \vee M \vee S$
$L \to C$
$(S \wedge M) \to \sim C$
Therefore, $C \to L$.

The argument is valid if and only if the following statement form is a tautology:
$((L \vee M \vee S) \wedge (L \to C) \wedge ((S \wedge M) \to \sim C)) \to (C \to L)$
Can this statement ever be false? If it is false, then
(1) $L \vee M \vee S$ *is true*
(2) $L \to C$ *is true*
(3) $(S \wedge M) \to \sim C$ *is true*
(4) $C \to L$ *is false.*

Statement 4 tells us that C is true and L is false. If C is true, then $\sim C$ *is false. Statement 3 is an implication that is true but its conclusion is false. Therefore, its hypothesis must be false also, so we conclude that* $S \wedge M$ *is false. Now, consider statement 1, which asserts that* $L \vee M \vee S$ *is true. We already know L is false, and that at least one of S and M is false. But S and M can't both be false. So, exactly one of S and M is false and the other is true.*
We have developed this scenario: Nigel saw a cricket match. He did not visit Leeds, but he did visit either Manchester or Southampton but not both. Therefore, all the premises are true but the conclusion is false. Therefore, the argument is invalid.

14. Consider the following statement.

"The sum of any two odd integers is even."

Write this statement using logic symbols, using variables and quantifiers in the following format:
"For all __________, if ___________ then ___________."

For all integers x and y, if x and y are odd, then x+y is even.

Alternatively, you could write:
$\forall x,y \in Z$, *(x and y are odd)* $\rightarrow$ *(x + y is even).*

15. How would we write the following statements and into logic symbols? You should draw your variables from the set of all animals.
    a. Some dogs chase all rabbits.

       $\exists x\ (D(x) \wedge \forall y\ (R(y) \rightarrow C(x,y)))$

    b. Only dogs chase rabbits.

       $\forall x\ (\forall y\ (R(y) \rightarrow C(x, y)) \rightarrow D(x))$

    c. All dogs chase rabbits and geese.

       $\forall x\ (D(x) \rightarrow \forall y((R(y) \vee G(y)) \rightarrow C(x, y)))$

    d. Only dogs chase rabbits and geese.

       $\forall x(\forall y((R(y) \vee G(y)) \rightarrow C(x, y)) \rightarrow D(x))$

    e. Some dogs chase only squirrels and rabbits.

       $\exists x\ (D(x) \wedge \forall y\ (C(x, y) \rightarrow (S(y) \vee R(y))))$

    f. All dogs chase some rabbits.

       $\forall x\ (D(x) \rightarrow \exists y\ (R(y) \wedge C(x, y)))$

    g. Only cats and dogs play inside and sleep outside.

       $\forall x\ ((P(x) \wedge S(x)) \rightarrow (C(x) \vee D(x)))$

    h. Some dogs and alligators chase only geese.

       $\exists x\ ((D(x) \vee A(x)) \wedge \forall y(C(x, y) \rightarrow G(y)))$

    i. No dog chases only cats and squirrels.

16. Write the negation of each of the statements in the previous question.
    a. Some dogs chase all rabbits.

All dogs do not chase some rabbits.

b. Only dogs chase rabbits.

All rabbits are chased by something that is not a dog.

c. All dogs chase rabbits and geese.

Some dogs do not chase both rabbits and geese.
OR: There are some dogs that do not chase rabbits or that do not chase geese.

d. Only dogs chase rabbits and geese.

All rabbits and geese are chased by something that is not a dog.

e. Some dogs chase only squirrels and rabbits.

All dogs chase something that is neither a squirrel nor a rabbit.

f. All dogs chase some rabbits.

Some dogs do not chase all rabbits.

g. Only cats and dogs play inside and sleep outside.

There exists an animal that plays inside and sleeps outside, but is neither a cat nor a dog.

17. Suppose S is a set containing some real numbers. Rewrite the following statement using logic symbols: "There is some number in S that is greater than all of the other numbers in S."

$\exists x\ (x \text{ in } S \wedge \forall y\ ((y \text{ in } S \wedge x \neq y) \rightarrow (x > y)))$

18. Give an example of a suitable predicate for each of the following situations:

a. "$\exists x, P(x)$" is true and "$\exists x, \sim P(x)$" is true.

(Assuming that x is a real number) $x > 0$
$\exists x, P(x)$ *is true because we can choose* $x = 1$.
$\exists x, \sim P(x)$ *is true because* $\sim P(x)$ *means* $x \leq 0$ *and we can choose* $x = 0$.

b. "$\exists x, P(x)$" is true and "$\exists x, \sim P(x)$" is false.

If $\exists x, \sim P(x)$ *is false, then consider its negation:* $\forall x, P(x)$ *must be true. If a predicate is always true, then of course it is at least sometimes true. So, it suffices to find a predicate that is always true. So, let's try* $x = x$.

c. "$\exists x, P(x)$" is false and "$\exists x, \sim P(x)$" is true.

If $\exists x, P(x)$ *is false, then its negation is true, which is* $\forall x, \sim P(x)$. *All we need is a predicate that is always false. Let's try* $x > x$.

d. "∃x, P(x)" is false and "∃x, ~P(x)" is false.

Let's turn both of these statements around to find out what is true: we want ∀x ~P(x) to be true, and ∀x P(x) to be true. Ordinarly, there is no way for a predicate to be true and false for all x. This is impossible. The only way this could work is if x is being drawn from an empty set. This is the case of vacuous truth.

19. Negate these statements:

a. For all integers n, n is prime or n is even.

There exists an integer n such that n is not prime and n is not even.

Note: it's safe to write "odd" in place of "not even" since every integer is either even or odd.

b. For all integers a and b, if a does not divide b, then a > b or b does not divide a.

There exist integers a and b for which: a does not divide b, and a ≤ b and b divides a.

c. For all real numbers x, there exists a real number y such that 0 < y < x.

There exists a real number x, such that for all real numbers y, 0 ≥ y or y ≥ x.

d. In some countries, all citizens can vote.

In all countries, there are citizens who cannot vote.

20. Consider the following statement: For all real numbers x, there exists a real number y such that xy = 4.

a. Write the negation of the statement.

There exists a real number x, such that for all real numbers y, xy ≠ 4.

b. Is the original statement true or false?

If we look at the negated statement, we see that it is true if we let x = 0. In this case, no matter what value of y we choose, xy is always 0, not 4. Therefore, the original statement is false.

21. Let S = { −3, −1, 3, 5, 7, 9 }. Consider the following statement.
For all x in S, if x is even, then x > 10.
Is this statement true or false? Justify your answer.

The statement is true due to vacuous truth. There is no even value of x, so we can say anything about such values.

22. Suppose F equals the XOR (exclusive or) of the Boolean variables x and y. Write a formula for F that uses inclusive OR, AND and/or NOT only, instead of XOR.

    XOR means either x or y is true, but not both.
    There's more than one way you could write your answer, for example
    F = x'y + xy' *or* *F = (x + y)(xy)'*

23. Let's practice DeMorgan's law.

    a. Re-write the Boolean function F(x,y,z) = (x + y' + z)' in terms of AND and NOT operations only. Then, re-write it again with only NAND operations (the symbol for NAND is the vertical bar |).

    Using DeMorgan's law, we can simplify F to become x'yz'.

    How do we write a negation using only NAND? It turns out that x' = <u>x|x</u>.
    Next, how do we do AND? xy = ((xy)')' = (x|y)' = <u>(x|y)|(x|y).</u>

    Let's first "and" the first two factors, x'y, and later "and" this result with z'.

    x' = x|x
    x'y = (x|x)y = ((x|x)|y)|((x|x)|y)

    z' = z|z
    (x'y)z = [((x|x)|y)|((x|x)|y)]z =
    ([((x|x)|y)|((x|x)|y)] | z) | ([((x|x)|y)|((x|x)|y)] | z)

    b. Re-write the Boolean function F(x, y) = x + y' in terms of AND and NOT operators.

    F = x + y'
    F' = (x + y')' = x'y *(by DeMorgan's law)*
    F = F'' = (x'y)'

    c. Re-write the Boolean function F(w, x, y, z) = w + x(y' + z) in terms of AND and NOT operators.

    y' + z = (yz')'
    In the expression w + x(yz')', we need to negate both the w and the x(yz')'.
    So, w + x(y' + z) = w + (yz')'
    = the negation of: w' ANDed with ((yz')')'
    = (w' ((yz')')')'

    d. Re-write the Boolean function F(x, y, z) = (x + y) (y' + z) in terms of AND and NOT operators.

Let's work on the two factors separately.

$(x + y) = ((x + y)')' = (x'y')'$

$(y' + z) = ((y' + z)')' = (yz')'$

Therefore, our answer is

$(x'y')' \ (yz')'$

e. Re-write the Boolean function F(x, y, z) = x(yz)' using only OR and NOT operators.

$F = x(yz)' = x(y' + z')$
$= xy' + xz'$
$= (x' + y)' + (x' + z)'$

f. Re-write the Boolean function F(x, y, z) = x(y + y'z) using only OR and NOT operators.

$F = x(y + y'z)$
$= xy + xy'z$
$= (x' + y')' + (x' + y + z')'$

24. Suppose we are designing digital circuits that have three binary inputs. How many distinct Boolean functions exist? Justify your answer.

3 inputs means $2^3 = 8$ row in the truth table. For each row, the functional result is either 0 or 1. So, the number of functions is 2^8 or 256.

25. Draw the digital circuit (using logic gates) corresponding to the boolean formula F = xy' + xyz.

Feed x, y, z into an AND gate.
Splice the input y, and feed y into an inverter.
Splice the input x, and feed x and the output of the inverter into a second AND gate.
Finally, feed the outputs of both AND gates into an OR gate.

26. For the following problems, design a digital circuit and simplify it using a Karnaugh map.
   a. We have 3 inputs and 1 output. The circuit should output 1 if at most one of the inputs equals 1, and it should output 0 otherwise.

   Among the numbers 0-7, which ones have all zeros or only a single 1 in its binary representation? It turns out they are 0, 1, 2, and 4. In other words, we need to simplify $F(x, y, z) = \Sigma(0, 1, 2, 4)$.

	yz = 00	*yz = 01*	*yz = 11*	*yz = 10*
x = 0	*1*	*1*		*1*
x = 1	*1*			

There are 3 terms based on which minterms can be combined:
000 and 100 → y'z'

000 and 001 → x'y'
000 and 010 → x'z'
So our answer is y'z' + x'y' + x'z'.

b. The bottom light in a single-digit display. Note that input values 10-15 are don't cares.

When is the bottom light lit? For these digits: 0, 2, 3, 5, 6, 8, 9. Our function to minimize is these minterms, plus the optional don't cares. The Karnaugh map becomes:

	yz = 00	*yz = 01*	*yz = 11*	*yz = 10*
wx = 00	*1*		*1*	*1*
wx = 01		*1*		*1*
wx = 11	*x*	*x*	*x*	*x*
wx = 10	*1*	*1*	*x*	*x*

We can combine minterms as follows:
The entire bottom half: w
Minterms that wrap around: 3,2,11,10: x'y
The 4th column: yz'
The 4 corners: x'z'
Minterms 5 and 13: xy'z

So, our answer is: w + x'y + yz' + x'z' + xy'z.

c. We have 4 inputs and 1 output. The output is true if and only if the binary value of the input, when interpreted as a 4-bit binary number, is less than our equal to 4.

Σ (0, 1, 2, 3, 4) = w'x'y'z' + w'x'y'z + w'x'yz' + w'x'yz + w'xy'z'

	yz = 00	*yz = 01*	*yz = 11*	*yz = 10*
wx = 00	*1*	*1*	*1*	*1*
wx = 01	*1*			
wx = 11				
wx = 10				

We can combine minterms as follows:
The first row: w'x'
The first two rows in the first column: w'y'z'

Our answer is w'x' + w'y'z' = w'(x' + y'z')

27. Consider the boolean function F (x, y, z) = (x XOR y) XOR z. Then, it is possible to express F in terms of only AND and OR operations, so that F comprises four minterms. Namely:

F = ____*xyz*_____ + ____*xy'z'*_____ + ____*x'yz'*____ + ____*x'y'z*____
Fill in the blanks to identify the minterms that make up F. Do not use minterm numbers.

If you work out the truth table, you will see that the only rows that are true are those where an odd number of input variables are true.

28. Suppose a Boolean function F(w, x, y, z) consists of minterms 4 and 14.
    a. Write out the Boolean expression for F.

       It helps to write out the binary for 4 and 14, which are 0100 and 1110.
       F = w'xy'z' + wxyz'.

    b. Which additional minterms would be necessary so that minterms 4 and 14 can be combined into the same term?

       Look at where 4 and 14 reside in the Karnaugh map. They are not in the same row or column. To combine, we also need at least minterms 6 and 12 to complete a set of four.

29. Suppose F is a function of five boolean variables a, b, c, d, e. The minterms are numbered from 0 to 31, where minterm 0 represents a'b'c'd'e' and minterm 31 represents abcde. Then, the four minterms 9, 11, 25 and 27 can combine to form what single term?

    First, convert the numbers to binary. Note that the 5 bits correspond to abcde:
    abcde
    9 → 01001
    11 → 01011
    25 → 11001
    27 → 11011

    Looking at each column of bits, what bit values do all 4 numbers share?
    b = 1, c = 0, c = 1. This means the term is bc'e.

30. How many minterms do you need to combine in order to create a term involving just a single variable such as w or y' if the function has:
    a. Three variables?

       Half of all squares: four

    b. Four variables?

       Half of all squares: eight

31. The following truth table defines a boolean function A. Derive the formula for A, using as few logical operators as you can.

P	q	r	A
T	T	T	F
T	T	F	T
T	F	T	F
T	F	F	T
F	T	T	F

F	T	F	F
F	F	T	F
F	F	F	T

	qr = 00	*01*	*11*	*10*
p = 0	*1*			
p = 1	*1*			*1*

Combining the first column: q'r'.
Combining the two bottom corners: pr'.
Our answer is pr' + q'r', but we can factor out r': r'(p + q').
We could even go one step further and write the or as an implication: r'(q → p).

32. Simplify these Boolean functions using a Karnaugh map:
    a. F(x,y,z) = Σ (0, 4, 5)

	yz = 00	*yz = 01*	*yz = 11*	*yz = 10*
x = 0	*1*			
x = 1	*1*	*1*		

One term combines 000 with 100, which is y'z'.
The other term combines 100 with 101, which is xy'.
So, our answer is y'z' + xy'.

b. F(w,x,y,z) = Σ (0, 4, 5, 6, 8, 9)

	yz = 00	*yz = 01*	*yz = 11*	*yz = 10*
wx = 00	*1*			
wx = 01	*1*	*1*		*1*
wx = 11				
wx = 10	*1*	*1*		

0000 + 0100 → w'y'z'
0000 + 1000 → x'y'z'
0100 + 0101 → w'xy'
1000 + 1001 → wx'y'
0100 + 0110 → w'xz'
Our final answer is w'y'z' + x'y'z' + w'xy' + wx'y' + w'xz'.

c. F(w,x,y,z) = Σ (0, 1, 3, 4, 5, 6, 7, 8, 9)

	yz = 00	*yz = 01*	*yz = 11*	*yz = 10*
wx = 00	*1*	*1*	*1*	
wx = 01	*1*	*1*	*1*	*1*
wx = 11				
wx = 10	*1*	*1*		

The second row: 0100 + 0101 + 0111 + 0110 → w'x
The group of 4 in the top center: 0001 + 0011 + 0101 + 0111 → w'z
The first two cells in the top & bottom rows: 0000 + 0001 + 1000 + 1001 → x'y'
Our final answer is w'x + w'z + x'y'

d. F(w,x,y,z) = Σ (2, 3, 6, 10, 11, 15)

	yz = 00	*yz = 01*	*yz = 11*	*yz = 10*
wx = 00			*1*	*1*
wx = 01				*1*
wx = 11			*1*	
wx = 10			*1*	*1*

Top right 2/bottom right 2: 0011 + 0010 + 1011 + 1010 → x'y
Top 2 in 4th column: 0010 + 0110 → w'yz'
Bottom 2 in 3rd column: 1111 + 1011 → wyz
Our final answer is x'y + w'yz' + wyz

e. Repeat part (c) but assume we also have don't cares = Σ (5, 7).

	yz = 00	*yz = 01*	*yz = 11*	*yz = 10*
wx = 00			*1*	*1*
wx = 01		*X*	*X*	*1*
wx = 11			*1*	
wx = 10			*1*	*1*

3rd column: yz
Upper right quadrant: w'y
Top right 2/bottom right 2: x'y
Our final answer is yz + w'y + x'y
Notice that the don't cares made our function simpler.

f.F(w,x,y,z) = Σ (1, 3, 7, 11, 15) with don't cares = Σ (0, 2, 5).

1 = 0001 0 = 0000
3 = 0011 2 = 0010
7 = 0111 5 = 0101
11 = 1011
15 = 1111

	yz = 00	01	11	10
wx = 00	X	1	1	X
01		X	1	

11			1	
10			1	

Top row: w′x′
Third column: yz
Total answer = w′x′ + yz

Alternatively: Top center: w′z
Third column: yz
Total answer = w′z + yz = z(w′ + y)

g. F(w,x,y,z) = Σ (3, 4, 14, 15) with don't cares = Σ (1, 6, 7, 10, 12).

Binary:
3 = 0011
4 = 0100
14 = 1110
15 = 1111
1 = 0001
6 = 0110
7 = 0111
10 = 1010
12 = 1100

	yz = 00	*01*	*11*	*10*
wx = 00		*x*	*1*	
wx = 01	*1*		*x*	*x*
wx = 11	*x*		*1*	*1*
wx = 10				*x*

We can combine 4 squares in the middle 2 rows that wrap around: xz′
We can combine 4 squares in the middle 2 rows on the right: xy
We can combine 2 squares in the top center: w′x′z
The sum of these 3 terms is our answer: F = xz′ + xy + w′x′z

h. F(w,x,y,z) = ∑ (0, 2, 5, 8, 13), with don't cares = ∑ (7, 10, 14, 15).

0 = 0000
2 = 0010
5 = 0101
8 = 1000
13 = 1101
7 = 0111
10 = 1010
14 = 1110
15 = 1111

	yz = 00	*01*	*11*	*10*
wx = 00	*1*			*1*
wx = 01		*1*	*x*	
wx = 11		*1*	*x*	*x*
wx = 10	*1*			*x*

We can combine the 4 squares in the middle: xz.
We can combine the 4 corners: x′z′.
The sum of these is our final answer: xz + x′z′.

Incidentally, our answer could also be written using exclusive or: ~(x XOR z).

i. F(w,x,y,z) = ∑ (1, 3, 9, 10, 11) with don't cares ∑ (2, 3, 7, 12).

	yz = 00	*01*	*11*	*10*
wx = 00		*<u>1</u>*	*<u>1.</u>*	*x.*
01	*x*		*x*	
11	*x*			
10		*<u>1</u>*	*<u>1.</u>*	*1.*

Combining the four minterms that are underlined: x′z
Combining the four minterms that are marked with a period: x′y
Our answer is x′y + x′z, and we can factor out an x′: x′(y + z).

33. Suppose x is a real number in the range $7 < x \le 10$. Write a ceiling or floor expression that rounds all these values of x, and only these values of x, to the integer 5.

	$7 < x \le 10$
Subtract 1 so the endpoints are multiples of 3:	*$6 < x - 1 \le 9$*
We want the endpoints to be consecutive int:	*$2 < (x - 1)/3 \le 3$*
We want the upper endpoint to be 5:	*$4 < 2 + (x - 1)/3 \le 5$*

The expression $2 + (x - 1)/3$ can be rewritten as $(x + 5)/3$. Thus, we want the ceiling of $(x+5)/3$.

34. The floor of $(x+7)/4$ equals 6 for which real numbers x?

    The floor means that we round down to 6. The expression $(x+7)/4$ is between 6 and 7, but could equal 6 exactly.

    $6 \le (x+7)/4 < 7$
    $24 \le x+7 < 28$
    $17 \le x < 21$

35. State the definition of "divides." In other words, the definition of the notation $a \mid b$.

    There exists an integer k such that: $ak = b$

36. Write direct proofs for the following:
    a. For all integers a, b, c and d, if $a \mid b$ and $c \mid d$ then $ac \mid (bc + ad)$.

       By the definition of "divides," there exist integers m and n such that:
       $am = b$ and $cn = d$.

       We need to show that $bc + ad$ is an integer multiple of ac.
       Since $b = am$, we can rewrite bc as amc.
       Since $d = cn$, we can rewrite ad as acn.
       Therefore, $bc + ad = amc + acn$, and we can factor out ac:
       $$bc + ad = ac(m + n)$$
       Thus, ac divides $bc + ad$.

    b. For all integers a, x and y, if $a \mid (x + y)$ and $a \mid (x - y)$, then $a \mid (5x + y)$.

       By the definition of "divides," there exist integers m and n such that:
       $am = (x + y)$ and $an = (x - y)$.

       We need to show that $5x + y$ is a multiple of a. We do this by expressing $5x + y$ in terms of the terms we already know about.

       Notice that $(x + y) + (x - y) = 2x$
       Therefore, $2((x + y) + (x - y)) = 4x$

       Also, notice that $5x + y = (x + y) + 4x$.
       Therefore, $5x + y = (x + y) + 2((x + y) + (x - y))$
       $$5x + y = 3(x + y) + 2(x - y)$$

And we can substitute from above:

$$5x + y = 3am + 2an$$
$$5x + y = a(3m + 2n)$$

And we note that 3m + 2n is an integer. This equation says that the number 5x + y equals a times some integer. Thus, a divides 5x + y.

c. If a, b, c and d are nonzero integers, and a | c and b | d, then ab | cd.

Since a | c and b | d, there exist integers p and q such that:
$ap = c$ and $bq = d$
Consider the number cd. We can substitute from the above equations:
$cd = (ap)(bq) = (ab)(pq)$
Note that pq is an integer. By the definition of divides, ab | cd, because we see there is an integer k such that abk = cd. That integer k equals pq.

d. The product of any two odd integers is odd.

Let the odd integers be x and y. By the definition of odd, there exist integers m and n such that:
$x = 2m + 1$
$y = 2n + 1.$
Consider the number xy:
$xy = (2m+1)(2n + 1) = 4mn + 2m + 2n + 1$
$\quad = 2(2mn + m + n) + 1$
And we observe that (2mn + m + n) is an integer.
Thus, xy satisfies the definition of odd.

e. If a, b and c are odd integers, then (2a – 4b + 5c) is also an odd integer.

Let a, b and c be odd integers. Then, by the definition of "odd" there exist integers p, q and r such that:
$a = 2p + 1, \quad b = 2q + 1, \quad c = 2r + 1$

Consider the number 2a – 4b + 5c. This number equals:
$2(2p + 1) - 4(2q + 1) + 5(2r + 1)$
$= 4p + 2 - 8q - 4 + 10r + 5$
$= 4p - 8q + 10r + 3$
$= 4p - 8q + 10r + 2 + 1$
$= 2(2p - 4q + 5r + 1) + 1$

Since p, q and r are integers and we obtain integers when adding, subtracting or multiplying them, 2p – 4q + 5r + 1 is also an integer.
So, by the definition of odd, 2a – 4b + 5c is odd, because it can be written in the form 2k+1, where in this case k = 2p – 4q + 5r + 1.

f. For all integers x and y, if x+y is even, then x and y are both even or both odd.

Let x and y be integers such that x + y is even. Since x + y is even, there exists an integer a such that x + y = 2a. Therefore, y = 2a – x. We will consider two cases. Either x is odd or even.
Case 1: Suppose x is odd. Then x = 2b + 1 for some integer b. If we substitute into the formula for y, we obtain y = 2a – (2b + 1) = 2a – 2b – 1 = 2(a – b – 1) + 1. Since a, b, and 1 are integers, then the number a – b – 1 is an integer also. Thus, y satisfies the definition of an odd number, and so x and y are both odd.

Case 2: Suppose x is even. Then x = 2c for some integer c. If we substitute into the formula for y, we obtain y = 2a – 2c = 2(a – c). Since a and c are integers, a – c is an integer also. Thus, y satisfies the definition of an even number, and so x and y are both even.

Therefore, x and y are both odd or both even.

g. For all integers n, $(3n^2 + 5n) / 2$ is an integer.

This is the same thing as saying that 2 divides $3n^2 + 5n$ *for all integers n.*

We can rewrite $3n^2 + 5n$ *as follows:*
$3n^2 + 5n = 3n^2 + 3n + 2n$
$\qquad\quad = 3n(n+1) + 2n$

Notice that the last expression has two terms, namely 3n(n+1) and 2n. Certainly, 2n is even for all integers n. But what can we say about 3n(n+1)? It is the product of the numbers 3, n and n+1. Notice that n and n+1 are consecutive integers. Thus either n or n+1 is even. Therefore n(n+1) is the product of an odd and an even integer, so it's even. And if you multiply an even integer by 3, it's still going to be even.

Now, we know that 3n(n+1) and 2n are even. When we add even numbers, the result is even. Thus, 3n(n+1) + 2n is even, so when we divide it by 2, the result is an integer.

(Note that you could also have proved this assertion by considering two cases, where n itself is odd, and where n is even.)

h. Let x and y be rational numbers where x < y. Also, let m = (x + y) / 2. Therefore, x < m.

We are given that:	$x < y$
Add x to both sides:	$2x < x + y$
Divide by 2:	$x < (x + y) / 2$
Definition of m:	$x < m$

37. The following statements are false. Show they are false by giving a counterexample.

a. For boolean variables A, B and C, if A → B is true, B → C is true, and A is false, then C is false.

We want to show that A → B and B → C can still be true without requiring that A and C both be false. To make this work, we need to specify values for all the variables. Let A = false, B = false and C = true. Then, A → B and B → C are still true.

b. The sum of two even numbers is an integer multiple of 4.

We need to come up with two even numbers whose sum is not a multiple of four. Let's choose our two numbers to be 4 and 2. Then, 6 is not divisible by 4.

c. The sum of two irrational numbers is irrational.

We need to come up with two irrational numbers whose sum is rational. The easiest way to do this is to pick some irrational number and its opposite. For example, $\sqrt{2}$ is irrational. Thus, we can let our two numbers be $\sqrt{2}$ and $-\sqrt{2}$. Their sum is zero, which is rational.

d. If n^2 mod 10 = 9 then n mod 10 = 3.

We need to produce a number n for which n^2 mod 10 equals 9, but n mod 3 is not 3. A number "mod 10" is essentially referring to the ones' digit. We need to think of a number that does not end in 3 but whose square ends in 9. Let n = 7, so that n^2 = 49. In this case, n^2 mod 10 = 9 but n mod 10 = 7.

38. Use proof by contradiction or proof by contraposition to show the following.

a. The empty set is unique.

We prove by contradiction. Suppose it is not unique. This means there exist two distinct empty sets. Let's call them E1 and E2. Since E1 is empty, then by vacuous truth, all of its elements are in E2. Thus, E1 is a subset of E2. By the same token, since E2 is empty, all of its elements are in E1. Thus, E2 is a subset of E1.

Since E1 and E2 are subsets of each other, this means they must be equal. But this contradicts our supposition that they were distinct.

Therefore, there can only be one empty set.

b. If ab is even, then a is even or b is even.

We prove by contraposition. The contrapositive of the given statement is: "if a is odd and b is odd, then ab is odd."

We proved earlier that the product of two odd integers is odd. You could repeat the proof here and you're done. ☺

c. If n is an integer and 3n + 2 is even, then n is even.

Suppose that statement were not true. Then its negation must be true, and we have the following information: 3n + 2 is even and n is odd.

Since n is odd, it can be written of the form (2k + 1), for some integer k.
Consider the number 3n + 2.
3n + 2 = 3(2k + 1) + 2 = 6k + 5 = 6k + 4 + 1 = 2(3k + 2) + 1

Note that 3k + 2 is an integer. Thus, 3n + 2 satisfies the definition of an odd number because it equals 2(integer) + 1. But this contradicts our earlier assumption that 3n + 2 is even. Having arrived at a contradiction, we must conclude that the original statement must hold.

d. If n is an integer and n^3 is odd, then n is odd.

Suppose not. Then we know that n is an integer, n^3 is odd, and n is even.
Since n is even, there exists an integer k such that n = 2k.
Consider the number n^3.
$n^3 = (2k)^3 = 8k^3 = 2(4k^3)$
Since k is an integer, $4k^3$ is also an integer.
So, by definition, n^3 is even, which contradicts our earlier assumption that n^3 is odd.
Therefore, the original statement must be true.

e. If x is irrational, and r is a rational number other than 0, then rx is irrational.

We prove by contradiction. Suppose the assertion did not hold. Then we have the following information: x is irrational, r is a nonzero rational number, and rx is rational.

Recall that when you divide rational numbers, the result is rational. In other words, a/b divided by c/d is ad/bc, as long as b, c and d are all nonzero.

Since rx is rational and r is a nonzero rational, we can divide these rational numbers to derive another rational number, which would equal x. However, x being rational contradicts the earlier fact that x is irrational.

Since we arrived at a contradiction, the original assertion must hold.

39. Prove or disprove:

a. There exists a real number x, such that for all real numbers y, x = y + 1.

This statement is false because its negation is true.

The negation reads as follows:
"For all real numbers x, there exists a real number y, such that $x \neq y + 1$."
To show this is true, we need to write a specific formula for y that works for any x.
Let x be any real number, and let y = x. Since x and y are equal, we can say that $x \neq y + 1$.

Thus, negation is true, so the original statement is false.

b. $\forall x \in Z, \exists y \in Z: y > x^2$

In English, this is saying that for all integers x, there is some integer y where $y > x^2$.
This is true. For any x, let $y = x^2 + 1$.
Then, for every x, $y = x^2 + 1 > x^2$ since 1 > 0.

c. $\exists x \in Z, \forall y \in Z: x < y^2$

In English, this is saying there is some integer x, such that for all integers y, $x < y^2$*.*
This is true. Notice that y^2 *is never negative. So, let* $x = -1$*.*
For any y, $-1 < y^2$*.*

d. $\exists x \in \mathbf{Z}, x < -1 \rightarrow x^2 < -1$

Because this statement begins "There exists…", in order to show it is true, we only need to find one value of x that works. The predicate is an implication. Notice that an implication is true when the hypothesis is false. When is $x < -1$ *false if we know x is an integer? Any integer from –1 or higher will make the hypothesis false. So, let's choose* $\underline{x = -1}$*. This makes the overall statement is true.*

e. $\forall x \in \mathbf{R}, \exists y \in \mathbf{R}, x - y > 5$

To be true, we need to be able to come up with a formula for y in terms of x, which will work no matter what the value of x is. Let's solve the inequality for y: $x - y > 5$ *implies that* $-y > 5 - x$*, or that* $y < x - 5$*. We need to select a value of y in terms of x such that y is guaranteed to be less than* $x - 5$*.* *Let's choose* $\underline{y = x - 6}$*.*
In this case, $x - y = x - (x - 6) = 6$*, which is certainly greater than 5. So, we have shown that the statement is true.*

f. For all real numbers x, there exists a real number y such that $0 < y < x$.

False, because the negation is true. We need just one value of x for which no value of y can satisfy the inequality. Let $x = 0$*. Then y has to be strictly between 0 and 0, which is impossible.*

g. For all real numbers x, there exists a real number y such that $y > x^2$.

True. For any real number x, let $y = x^2 + 1$*. Then* $y > x^2$*, because this is equivalent to* $x^2 + 1 > x^2$*, which in turn is equivalent to* $1 > 0$*.*

40. Write either a direct or indirect proof of this statement:
For all integers x and y, if $x^4 + 4y^4$ is odd, then $x^2 + 2xy + 2y^2$ is also odd.

Direct proof
We are given that $x^4 + 4y^4$ *is odd. Notice that* $4y^4$ *is even because it is a multiple of 4. If we subtract an even number from an odd number, the result is an odd number. Therefore* x^4 *is an odd number.*

Let's recall the facts that if you multiply odd numbers the answer is odd, and if you multiply even numbers, the result is even. Is x itself odd or even? If x were even, x^4 *would also be* *even. But we have determined up to this point that x^4 is odd. So, x cannot be even, so it must be odd.*

Now, we know that x is odd, but we do not know if y is odd or even.
Let's examine the expression $x^2 + 2xy + 2y^2$. Since x is odd, x^2 is odd. The other two terms are clearly even because they are both 2 multiplied by some integer. If you add an odd plus two even numbers, the result is odd. Thus the expression $x^2 + 2xy + 2y^2$ is odd, and we are done.

<u>Indirect proof</u>
Suppose the statement were not true. Then there exist integers x and y for which $x^4 + 4y^4$ is odd but $x^2 + 2xy + 2y^2$ is even. Notice that the terms 2xy and $2y^2$ both represent even numbers because they are 2 times an integer. If we subtract them from $x^2 + 2xy + 2y^2$, we obtain x^2, and this has to be even since subtracting even integers yields another even integer. Since x^2 is even, we can square it and obtain another even integer, x^4. Finally, consider the number $x^4 + 4y^4$. We already know the first term is even. But the second term is also even because it is 4 times another integer. The sum of evens is even. Therefore, $x^4 + 4y^4$ is even, which contradicts our assumption that it was odd. Thus, the original assertion must hold.

<u>Indirect proof #2</u>
Suppose the statement were not true. Then there exist integers x and y for which $x^4 + 4y^4$ is odd but $x^2 + 2xy + 2y^2$ is even. Note that we can factor $x^4 + 4y^4$ as follows:
$x^4 + 4y^4 = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2)$
Since $x^2 + 2xy + 2y^2$ is even, we can multiply it by an integer such as $(x^2 - 2xy + 2y^2)$ to obtain another even number. Therefore, $x^4 + 4y^4$ is even, which contradicts our assumption that it was odd. Thus, the original assertion must hold.

41. What is the value of this arithmetical expression?

$$\sum_{i=1}^{3} 2^i \binom{3}{i}$$

2^1 C(3, 1) + 2^2 C(3, 2) + 2^3 C(3, 3)
= 6 + 12 + 8
= 26

42. Use induction to prove the following statements.
    a. For all positive integers n,
    (1)(2) + (2)(4) + (3)(6) + (4)(8) + … + (n)(2n) = n(n+1)(2n+1)/3

    *Base case: let n = 1. Then the statement becomes (1)(2) = 1(1+1)(2*1+1)/3, which simplifies to 2 = 1*2*3/3, or 2=2, which is true.*

    Inductive case: Suppose the statement true for n = k. That is,
    (1)(2) + (2)(4) + (3)(6) + (4)(8) + … + (k)(2k) = k(k+1)(2k+1)/3

    Add the next term to both sides, which is (k+1)(2k+2):

    (1)(2) + (2)(4) + (3)(6) + (4)(8) + … + (k)(2k) + (k+1)(2k+2)

= k(k+1)(2k+1)/3 + (k+1)(2k+2)

We can simplify the right side as follows:
k(k+1)(2k+1)/3 + (k+1)(2k+2) = (1/3)(k+1) [k(2k+1) + 3(2k+2)]
= (1/3)(k+1) (2k^2 *+ k + 6k + 6]*
= (1/3)(k+1) (2k^2 *+ 7k + 6)*
= (1/3) (k+1) (k+2) (2k + 3)

This last expression is exactly what we would expect if we plug k+1 into the summation formula at the beginning in place of n.

Since P(1) is true, and P(k) → P(k+1), then P(n) is true for all positive integers n.

b. For all positive integers n, 4 + 8 + 12 + 16 + … + 4n = 2n(n+1).

*Base case: Let n = 1. P(1) says 4 = 2*1(1+1). This simplifies to 4 = 4, which is true.*

Inductive case: Assume that P(k) is true. That is, assume that for some k:
4 + 8 + 12 + 16 + … + 4k = 2k(k + 1)

Let's add the next term to both sides. The next term is 4(k+1) = 4k + 4:

4 + 8 + 12 + … + 4k + (4k + 4) = 2k(k + 1) + (4k + 4)

We simplify the right side of the equation as follows:
2k(k + 1) + 4k + 4
= 2k^2 *+ 2k + 4k + 4*
= 2k^2 *+ 6k + 4*
= 2(k^2 *+ 3k + 2)*
= 2(k+1)(k+2)

P(k+1) is a summation whose formula is 2(k+1)(k+2). So, we have achieved P(k+1).

Since P(1) is true, and P(k) → P(k+1), we can say that P(n) is true for all positive integers n.

c. For all integers n ≥ 0, $1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^n}=2-\frac{1}{2^n}$

The base case is where n = 0. If n = 0, the equation becomes

$$\frac{1}{2^0}= 2-\frac{1}{2^0}$$

This simplifies to 1 = 2 – 1, which is true.

Next, assume that the given equation true for some arbitrary integer k ≥ 0. That is, assume that:

$$1+\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2^k}=2-\frac{1}{2^k}$$

Let's add the next term of the series to both sides of the equation, and then simplify the right side:

$$1 + \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2^k} + \frac{1}{2^{k+1}} = 2 - \frac{1}{2^k} + \frac{1}{2^{k+1}}$$

$$= 2 - \frac{2}{2^{k+1}} + \frac{1}{2^{k+1}}$$

$$= 2 - \frac{1}{2^{k+1}}$$

This last equation is exactly what we would obtain if we substitute k+1 for n in the original equation. Therefore, we have obtained P(k+1).

Since P(0) is true, and P(k) implies P(k+1), the equation is true for all n ≥ 0.

d. For all positive integers n,

$$\sum_{i=1}^{n}(i^2 + 2i - 1)2^i = n^2 2^{n+1}$$

Base case: Let n = 1. Then the statement becomes $(1^2 + 2*1 - 1)2^1 = 1^2\ 2^{1+1}$*, which simplifies to (1 + 2 – 1)2 = 4, or 4 = 4, which is true.*

Inductive case. Suppose the statement is true for n = k. In other words, assume that the sum of the first k terms of the series equals $k^2\ 2^{k+1}$*. We have to show that the sum of the first k+1 terms equals* $(k+1)^2\ 2^{k+2}$*.*

Starting with the summation of the first k terms, we add the next term to the summation formula. The next term is k+1 substituted in for n in the term formula given on the left side of the original equation.

The right side of the equation becomes:
$k^2\ 2^{k+1} + ((k+1)^2 + 2(k+1) - 1)2^{k+1}$
$= 2^{k+1}\ (k^2 + (k+1)^2 + 2(k+1) - 1)$
$= 2^{k+1}\ (k^2 + k^2 + 2k + 1 + 2k + 2 - 1)$
$= 2^{k+1}\ (2k^2 + 4k + 2)$
$= 2^{k+1}\ 2(k^2 + 2k + 1)$
$= 2^{k+1}\ 2(k+1)^2$
$= (k+1)^2\ 2^{k+2}$

So, we have correctly derived the formula for the first k+1 terms.
Since, P(1) is true and P(k) → P(k+1), we can say that P(n) is true for all positive integers n.

e. For all integers n ≥ 7, $n^2 > 5n + 10$.

Base case: Let n = 7. P(7) is the statement that: $7^2 > 5*7 + 10$*.*
This simplifies to 49 > 45, which is true.

Inductive case. Suppose P(k) is true. In other words, assume that there exists some k ≥ 7 for which: $k^2 > 5k + 10$*.* *[A]*

We need to show P(k+1), i.e. we need to show that $(k+1)^2 > 5(k+1) + 10$.

Consider the difference between these inequalities. It is:

$(k+1)^2 - k^2 > [5(k+1) + 10\ ] - (5k + 10)$ [B]

$2k + 1 > 5k + 5 + 10 - 5k - 10$

$2k + 1 > 5$

$2k > 4$

$k > 2$

This last inequality is clearly true since we were given $k \geq 7$*. And this true inequality is logically equivalent to [B], the one we considered. Now, we are going to add inequalities [A] and [B] to obtain:* $(k+1)^2 > 5(k+1) + 10$*, and this is P(k+1).*

Since P(7) is true and P(k) $\rightarrow$ *P(k+1), then P(n) is true for all* $n \geq 7$*.*

f. For all positive integers n, $8 \mid (3^{2n} + 7)$.

Base case: let n = 1. Then, the expression $3^{2n} + 7$ *equals* $3^2 + 7 = 16$*, which is divisible by 8. So, P(1) is true.*

Inductive case: next assume that P(k) is true. That is, assume that $8 \mid (3^{2k} + 7)$*. We need to show that* $8 \mid (3^{2(k+1)} + 7)$*.*

Consider the difference between $3^{2k} + 7$ *and* $3^{2(k+1)} + 7$*. This works out to:*

$3^{2(k+1)} + 7 - (3^{2k} + 7)$

$= 3^{2(k+1)} - 3^{2k}$

$= 3^{2k+2} - 3^{2k}$

$= 3^{2k}(3^2 - 1)$

$= 8 * 3^{2k}$

The number $8 * 3^{2k}$ *is clearly a multiple of 8.*
Since 8 divides $3^{2k} + 7$*, and 8 divides* $3^{2(k+1)} + 7 - (3^{2k} + 7)$*, then 8 must also divide their sum, which is* $3^{2(k+1)} + 7$*. Therefore, P(k+1) is true.*

Since P(1) is true, and P(k) $\rightarrow$ *P(k+1), then P(n) is true for all positive integers n.*

g. For all integers $n \geq 0$, $10^n + 3(4^{n+2}) + 5$ is divisible by 9.

Base case: Let n = 0. Then, the expression $10^n + 3(4^{n+2}) + 5$ *becomes* $10^0 + 3(4^{0+2}) + 5$*, which equals* $1 + 3*16 + 5 = 54$*. Since* $9 \mid 54$*, we say that P(0) is true.*

Inductive case: Suppose P(k) is true for some nonnegative integer k. In other words, assume that the number $10^k + 3(4^{k+2}) + 5$ *is a multiple of 9. We need to show P(k+1), i.e. that* $10^{k+1} + 3(4^{k+3}) + 5$ *is also a multiple of 9.*

Consider the difference between these numbers, which is:

$10^{k+1} + 3(4^{k+3}) + 5 - [\ 10^k + 3(4^{k+2}) + 5\ \ ]$

$= 10^{k+1} - 10^k + 3(4^{k+3}) - 3(4^{k+2})$

$= 10^k(10 - 1) + 3(4^{k+2})(4 - 1)$

$= 9(10^k + 4^{k+2})$

Clearly, this number is divisible by 9. Since 9 divides both the number $10^k + 3(4^{k+2}) + 5$ *and the number* $10^{k+1} + 3(4^{k+3}) + 5 - [\ 10^k + 3(4^{k+2}) + 5\ ]$*, it must also divide their sum, which is* $10^{k+1} + 3(4^{k+3}) + 5$*. Thus, we have arrived at P(k+1).*

Since P(0) is true and P(k) → P(k+1), P(n) is true for all nonnegative integers n.

h. For all integers $n \geq 4$, $n^2 > n + 9$.

Base case: Let n = 4. P(4) says $4^2 > 4 + 9$*, and this simplifies to 16 > 13, which is true.*

Inductive case: Assume P(k) is true. That is, assume that for some k, $k^2 > k + 9$*. [A]*

We need to show that P(k+1) is true, which says that $(k+1)^2 > (k+1) + 9$*.*

Consider the difference between these inequalities. It is:

$(k + 1)^2 - k^2 > [(k + 1) + 9] - [k + 9]$ *[B]*

$2k + 1 > 1$

$2k > 0$

$k > 0$

Since $k \geq 4$*, it is clear that k > 0 as well. And this inequality is equivalent to the one we first considered, [B].*

Now, let's combine inequalities [A] and [B] by adding. We obtain:

$(k+1)^2 > (k+1) + 9$

And this is P(k+1).

Since P(4) is true, and P(k) → P(k+1), P(n) is true for all $n \geq 4$*.*

i. For all integers $n \geq 6$, $n^2 > 4n + 9$.

Base case: Let n = 6. P(6) says $6^2 > 4*6 + 9$*, and this simplifies to 36 > 33, which is true.*

Inductive case: Assume P(k) is true. That is, assume that for some k, $k^2 > 4k + 9$*. [A]*

We need to show that P(k+1) is true, which says that $(k+1)^2 > 4(k+1) + 9$*.*

Consider the difference between these inequalities. It is:

$(k + 1)^2 - k^2 > [4(k + 1) + 9] - [4k + 9]$ *[B]*

$2k + 1 > 4$

$2k > 3$

$k > 3/2$

Since $k \geq 6$*, it is clear that k > 3/2 as well. And this inequality is equivalent to the one we first considered, [B].*

Now, let's combine inequalities [A] and [B] by adding. We obtain:

$(k+1)^2 > 4(k+1) + 9$

And this is P(k+1).

Since P(6) is true, and P(k) → P(k+1), P(n) is true for all n ≥ 6.

j. For all integers $n \geq 4$, $n^3 \geq 7n + 12$.

The base case says that $4^3 \geq 7(4) + 12$. This simplifies to $64 \geq 40$, which is true.

Suppose the statement is true for n = k. In other words, assume that $k \geq 4$ and $k^3 \geq 7k + 12$. *(inequality #1)*

Our goal is to show that P(k+1) is true. In other words, we must show that $(k+1)^3 \geq 7(k+1) + 12$.

Consider the difference of these inequalities. It is:
$(k+1)^3 - k^3 \geq 7(k+1) + 12 - (7k + 12)$ *(inequality #2)*

We can simplify the above inequality as follows.
$k^3 + 3k^2 + 3k + 1 - k^3 \geq 7k + 7 + 12 - 7k - 12$
$3k^2 + 3k + 1 \geq 7$
$3k^2 + 3k \geq 6$
$k^2 + k \geq 2$

Since we already know that $k \geq 4$, the smallest possible value for the left side of the inequality is 20. Since $20 \geq 2$ and $k^2 + k \geq 20$, by transitivity of inequalities, we can now say $k^2 + k \geq 2$. And this inequality is equivalent to the one labelled inequality #2.

If we combine inequalities #1 and #2 by adding, we obtain P(k+1).

Since P(4) is true and P(k) → P(k+1), the original statement is true for all $n \geq 4$.

k. $n! < n^n$ for all integers $n \geq 2$.

Base case: Let n = 2. Then, the inequality says $2! < 2^2$. This simplifies to $2 < 4$, which is true.

Inductive case: Suppose P(k) is true for some $k \geq 2$. That is, assume that $k! < k^k$. We need to show P(k+1), i.e. that $(k+1)! < (k+1)^{k+1}$. *[A]*

Consider the quotient of these inequalities:

$(k+1)! / k! < (k+1)^{k+1} / k^k$ *[B]*
$k + 1 < (k+1)(k+1)^k / k^k$
$1 < (k+1)^k / k^k$
$1 < ((k+1)/k)^k$
$1 < (k+1)/k$
$1 < 1 + 1/k$
$0 < 1/k$

This last inequality is clearly true because we already knew that $k \geq 2$. And, this inequality is equivalent to the inequality we considered, labeled above as [B].

Finally, let's combine inequalities [A] and [B] by multiplying them together. Since all the members are positive, the direction of the inequality will not change. Multiplying, we obtain:

k! (k+1)! / k! < k^k $(k+1)^{k+1}$ / k^k
(k+1)! < $(k+1)^{k+1}$

And this last inequality is P(k+1).
Since P(2) is true, and P(k) → P(k+1), P(n) is true for all n ≥ 2.

l. For all integers n ≥ 5, $2^n > n^2$.

The base case is where n = 5. The inequality becomes $2^5 > 5^2$, which simplifies to 32 > 25, which is true.

Assume the inequality is true for some arbitrary integer k ≥ 5. That is, assume that P(k) is true, and let's call this inequality #1:

$$2^k > k^2$$

We wish to show that P(k+1) is true. In other words, our goal is to show that:

$$2^{k+1} > (k+1)^2$$

Consider the quotient of these inequalities, and call it inequality #2. It is:

$$\frac{2^{k+1}}{2^k} > \frac{(k+1)^2}{k^2}$$

This inequality is equivalent to:

$$2 > \frac{k^2 + 2k + 1}{k^2}$$

$$2 > 1 + \frac{2}{k} + \frac{1}{k^2}$$

$$1 > \frac{2}{k} + \frac{1}{k^2}$$

Since k ≥ 5, 2/k cannot be larger than 2/5, and $1/k^2$ cannot be larger than 1/25. Therefore, the right side of the inequality cannot be larger than 0.44. Therefore, this inequality is clearly true, and it is equivalent to inequality #2.

Since inequalities #1 and #2 are true we can multiply them to obtain:

$$2^k \; \frac{2^{k+1}}{2^k} > k^2 \; \frac{(k+1)^2}{k^2}$$

which becomes:

$$2^{k+1} > (k+1)^2$$

And this inequality is P(k+1).

Since P(5) is true and P(k) implies P(k+1), P(n) is true for all n ≥ 5.

m. Every positive integer is either even or odd.

Base case: Let n = 1. The number 1 is odd. Thus, the statement P(1) is true.

Inductive case: Assume that P(k) is true. That is, assume that the integer k is even or odd. We need to show that k+1 is even or odd.

We need to consider two cases:
First, if k is even, then it can be written of the form k = 2a for some integer a. Therefore, k+1 = 2a+1, and we see that k+1 satisfies the definition of an odd number.

Second, if k is odd, then it can be written of the form k = 2a+1 for some integer a. Therefore, k+1 = 2a+2 = 2(a+1). And here we see that k+1 satisfies the definition of an even number.

In either case, k+1 is even or odd, so we see that P(k+1) is true.

Since P(1) is true, and P(k) → P(k+1), then P(n) is true for all positive integers n.

n. $3^{4n+2} + 5^{2n+1}$ is divisible by 14 for all positive integers n.

Base case: Let n = 1. The expression $3^{4n+2} + 5^{2n+1}$ *when n = 1 simplifies to* $3^6 + 5^3 = 729 + 125 = 854$. *This number is a multiple of 14 because 14*61 = 854.*

Inductive case: Assume P(k) is true. In other words, assume that $14 \mid 3^{4k+2} + 5^{2k+1}$ *for some positive integer k. We need to show that* $14 \mid 3^{4k+6} + 5^{2k+3}$.

It turns out that my subtraction technique doesn't work for this problem, so let's carefully rewrite f(k+1) in terms of f(k) and a term obviously divisible by 14.

$$\begin{aligned} 3^{4k+6} + 5^{2k+3} &= 81 * 3^{4k+2} + 25 * 5^{2k+1} \qquad \text{(and note that } 81 = 56+25\text{)} \\ &= 56 * 3^{4k+2} + 25 * 3^{4k+2} + 25 * 5^{2k+1} \\ &= 14\,(4 * 3^{4k+2}) + 25\,(3^{4k+2} + 5^{2k+1}) \end{aligned}$$

This last expression has two terms. The first term is clearly a multiple of 14. The second one is also a multiple of 14 because we had already assumed that $14 \mid 3^{4k+2} + 5^{2k+1}$. *Therefore,* $14 \mid 3^{4k+6} + 5^{2k+3}$, *and this is P(k+1).*

Since P(1) is true and P(k) → P(k+1), P(n) is true for all positive integers n.

o. For all positive integers n, 21 divides $4^{n+1} + 5^{2n-1}$.

The base case says that 21 divides $4^{1+1} + 5^{2(1)-1}$. *This number simplifies to 16 + 5 = 21. Therefore, the base case says 21 | 21, which is true.*

Assume that P(k) is true. In other words, assume that 21 divides $4^{k+1} + 5^{2k-1}$.
We need to show that P(k+1) is true, i.e. that 21 divides $4^{k+2} + 5^{2k+1}$

Consider the number $4^{k+2} + 5^{2k+1}$.

$$\begin{aligned} 4^{k+2} + 5^{2k+1} &= 4\,(4^{k+1}) + 25\,(5^{2k-1}) \\ &= 4\,(4^{k+1}) + 4\,(5^{2k-1}) + 21\,(5^{2k-1}) \\ &= 4\,(4^{k+1} + 5^{2k-1}) + 21\,(5^{2k-1}) \end{aligned}$$

This last expression has two terms. The first term is divisible by 21 because the number in parentheses was already assumed to be divisible by 21. The second term is clearly divisible by

21 because we literally have a factor of 21 (and it is being multiplied by an integer). Therefore, $4^{k+2} + 5^{2k+1}$ is divisible by 21, and this statement is P(k+1).

Since P(1) is true and P(k) → P(k+1), then P(n) is true for all positive integers n.

p. Any postage amount of 35 cents or more can be accomplished using 5- and 9-cent stamps.

This problem is basically saying that for all integers $n \geq 35$, there exist nonnegative integers x and y such that $n = 5x + 9y$.

Base case: Let $n = 35$. We need to find specific values of x and y such that $5x + 9y = 35$. Choose $x = 7$ and $y = 0$.

Inductive case: Assume that for some $k \geq 35$, $k = 5x + 9y$. We need to write a formula for k+1 where the 5 and 9 are multiplied by nonnegative integers.

To motivate our answer, let's review our solution for 35, and also solve for 36 and 37:
$35 = 5(7) + 9(0)$
$36 = 5(0) + 9(4)$ *notice that x decreased by 7 and y increased 4*
$37 = 5(2) + 9(3)$ *notice that x increased by 2 and y decreased by 1*

There are two alternative formulas for k+1:
$k + 1 = 5(x - 7) + 9(y + 4)$ *if* $x \geq 7$
$k + 1 = 5(x + 2) + 9(y - 1)$ *otherwise*

Therefore, P(k+1) is true.
Since P(35) is true, and P(k) → P(k+1), P(n) is true for all $n \geq 35$.

q. Repeat the previous part, but using the numbers 44, 5, and 12, respectively.

The problem is now saying that for all integers $n \geq 44$, there exist nonnegative integers x and y such that $n = 5x + 12y$.

Base case: Let $n = 44$. In this case, choose $x = 4$ and $y = 2$.

Inductive case: Assume that for some $k \geq 44$, $k = 5x + 12y$. We need to write a formula for k+1 where the 5 and 12 are multiplied by nonnegative integers.

To motivate our answer, let's review our solution for 44 and also solve for 45 and 46:
$44 = 5(4) + 12(2)$
$45 = 5(9) + 12(0)$ *notice that x increased by 5 and y decreased by 2*
$46 = 5(2) + 12(3)$ *notice that x decreased by 7 and y increased by 3*

We are ready to write our two alternative formulas for k+1:
$k + 1 = 5(x + 5) + 12(y - 2)$ *if* $y \geq 2$
$k + 1 = 5(x - 7) + 12(y + 3)$ *otherwise*

Therefore, P(k+1) is true.
Since P(44) is true, and P(k) → P(k+1), P(n) is true for all $n \geq 44$.

43. Consider the following code. How many times does the letter 'a' get printed?

```
for (i = 1; i <= 10; ++i)
   for (j = 1; j <= i*i; ++j)
      System.out.print("a");
```

We want the sum of i = 1 to 10 of the sum of j = 1 to i^2 of 1. The summand is just "1" because on each iteration we print a single letter a.

For the inner summation, we use the Bernoulli formula for 1, and note that the upper limit is i^2 rather than n. The Bernoulli formula says that the sum for j = 1 to n of 1 is n. Therefore, the sum for j = 1 to i^2 of 1 is i^2.

*Next, we need to evaluate the outer summation. It now says we have the sum for i equals 1 to 10 of i^2. This is a straightforward application of the i^2 Bernoulli formula. This formula is n(n+1)(2n+1)/6, and we plug in 10 for n here: 10(11)(21)/6 = 5(11)(7) = 35*11 = 385.*

44. Use Bernoulli formulas to determine the following.

a. If the nth term of a sequence is $2n^2 + 5n$, then what is the sum of the first n terms?

The sum is 2(the sum of i^2) + 5(the sum of i)
= 2n(n+1)(2n+1)/6 + 5n(n+1)/2

Which could be simplified as follows
= n(n+1)(2n+1)/3 + 5n(n+1)/2
= (1/6)n(n+1) [2(2n+1) + 3(5)]
= n(n+1)(4n+17)/6

b. If the sum of the first n terms of a series is $2n^2 + 5n$, then write a formula for the nth term.

The term formula can be calculated from the sum formula S this way: S(n) – S(n – 1). So, substitute n and n – 1 in the given summation formula, and subtract.

$$\begin{aligned} S(n) - S(n-1) &= 2n^2 + 5n - (2(n-1)^2 + 5(n-1)) \\ &= 2n^2 + 5n - (2(n^2 - 2n + 1) + 5n - 5) \\ &= 2n^2 + 5n - (2n^2 - 4n + 2 + 5n - 5) \\ &= 2n^2 + 5n - (2n^2 + n - 3) \\ &= 4n + 3 \end{aligned}$$

Therefore, our term formula is 4n + 3.
You could actually use Bernoulli formulas to check your answer.
4(n(n+1)/2) + 3(n) = 2n(n+1) + 3n = $2n^2$ + 5n.

c. What is the sum of the first n terms of a series whose n^{th} term is given by the formula $8n^3 + 6n^2$?

$$\sum_{i=1}^{n}(8\,i^3 + 6\,i^2) = 8\sum_{i=1}^{n} i^3 + 6\sum_{i=1}^{n} i^2 = 8\ \frac{n^2(n+1)^2}{4} + 6\ \frac{n(n+1)(2n+1)}{6}$$

$$= 2\,n^2(n+1)^2 + n(n+1)(2n+1)$$

45. Use Bernoulli formulas to simplify $50^2 + 51^2 + 52^2 + \ldots + 100^2$.

This is the sum from 1 to 100 minus the sum from 1 to 49. In both cases we want to sum i^2. So, all you have to do is plug in the numbers 100 and 49 in the expression n(n+1)(2n+1)/6 and subtract these two answers. The arithmetic is not interesting.

100(101)(201) / 6 – 49(50)(99) / 6
= 50(101)(67) – 49(25)(33)
= 297925.

46. Suppose that the following sum of consecutive integers from a through b, inclusive:

a + (a + 1) + (a + 2) + … + b

equals $\frac{83(84)}{2} - \frac{56(57)}{2}$.

What are the values of a and b?

(sum of 1 through 83) – (sum of 1 through 56) = (sum of 57 through 83), so
a = 57 and b = 83.

47. Consider the following code. Assume that n is a positive number.

```
sum = 0;
for (i = 1; i <= n; ++i)
    sum += (i + 4) * (i - 4);
```

a. Use Bernoulli formulas to determine the value of sum after the loop is finished.

We want the sum from i = 1 to n of (i + 4)(i – 4). The summand simplifies to (i^2 – 16).

The sum is n(n+1)(2n+1) / 6 – 16n
= (1/6)n [(n+1)(2n+1) – 96]
= (1/6)n ($2n^2$ + 3n – 95)
= n($2n^2$ + 3n – 95) / 6

b. A loop invariant property of this loop is that after k iterations, the value of sum will be whatever you get if you substitute k in for n in the summation formula you just found in part a. Show that this loop invariant property is satisfied.

Suppose that after k iterations the sum is k($2k^2$ + 3k – 95) / 6. The next term in the series is (k + 5)(k – 3). After adding this term we obtain:

k($2k^2$ + 3k – 95) / 6 + (k + 5)(k – 3)
= k($2k^2$ + 3k – 95) / 6 + k^2 + 2k – 15
= (1/6)($2k^3$ + $3k^2$ – 95k + $6k^2$ + 12k – 90)

= (1/6)($2k^3 + 9k^2 - 83k - 90$)

Hopefully, k + 1 is a factor of $2k^3 + 9k^2 - 83k - 90$. Let's use synthetic substitution to find out:

```
–1 /  2      9        –83     –90
     –2      –7       90
------------------------------
       2      7        –90     0
```

Since the remainder is 0, we see that the polynomial $2k^3 + 9k^2 - 83k - 90$ can be factored as $(k+1)(2k^2 + 7k - 90)$.

Therefore, according to our P(k) assumption, our formula for the first k+1 terms becomes $(k+1)(2k^2 + 7k - 90) / 6$.

We need to show that we would get the same thing if we plug k+1 into the original sum formula. That would say: $(k+1)(2(k+1)^2 + 3(k+1) - 95)/6$, and this works out to:

$(k+1)(2k^2 + 4k + 2 + 3k + 3 - 95) / 6$
$= (k+1)(2k^2 + 7k - 90) / 6$.

Yes, the two expressions match, so we have obtained P(k+1) from P(k).

c. How many operations are performed when the code executes?

The statements "sum = 0" and "i = 1" are performed once.
*The five operations <=, *, +, – and += are performed on every iteration.*
The ++ is performed on every iteration, plus one because of the need to exit the loop.

So, if we let L be the number of iterations, the total number of operations is $2 + 5L + (L+1) = 6L + 3$.

Since $L = n(2n^2 + 3n - 95) / 6$, we plug in the above expression:

 # operations $= 6\,[n(2n^2 + 3n - 95) / 6\,] + 3$
$= n(2n^2 + 3n - 95) + 3$
$= 2n^3 + 3n^2 - 95n + 3$

48. True or false… For any sets A, B and C: A – (B – C) = (A – B) – C. And explain how you arrived at your answer. (Hint: the most elegant way is to use a Karnaugh map. Or, you could re-write this statement about sets as a statement from propositional logic, and then see if the two expressions are logically equivalent.)

If we convert this to a logic statement, it would read:
A ^ ~(B ^ ~C) = (A ^ ~B) ^ ~C
A ^ (~B or C) = A ^ ~B ^ ~C
(A ^ ~B) or (A ^ C) = A ^ ~B ^ ~C

There is only one combination of values that makes the right side true. It is A=T, B=F, C=F. But the left side could be true if A=T and C=T. So, they are not equivalent.

49. Given that (A − C) ∪ (B − D) ⊆ (A ∪ B) − (C ∪ D) for sets A, B, C and D, then what set intersection(s) must be empty?

Let's convert each side of the ⊆ to logic, and use a Karnaugh map for each.

The first set is (A − C) ∪ (B − D). As a Boolean function, it would look like this:
F(A,B,C,D) = AC′ + BD′.
The second set is (A ∪ B) − (C ∪ D). As a Boolean function, it would say:
G(A,B,C,D) = (A+B)(C+D)′
= (A+B)(C′D′)
= AC′D′ + BC′D′

Here are the Karnaugh maps for F and G:

Function F					*Function G*				
	CD=00	*01*	*11*	*10*		*CD=00*	*01*	*11*	*10*
AB=00					*AB=00*				
01	*1*			*1*	*01*	*1*			
11	*1*	*1*		*1*	*11*	*1*			
10	*1*	*1*			*10*	*1*			

In order for F to be a subset of G, everything in F must be in G. We see inside the Karnaugh map of F, that there are some 1's that are not present in the Karnaugh map of G. These are the subsets that must be empty. Namely, these:
1101 = ABC′D
1001 = AB′C′D
0110 = A′BCD′
1110 = ABCD′

50. In a survey of 100 people, 35 said that they liked Bach, 15 said they liked both Bach and Mozart, while 20 people said they liked neither Bach nor Mozart. How many people liked Mozart?

We can set this up as a "2-variable" Karnaugh map:

	Likes Mozart	*Doesn't like Mozart*
Likes Bach	*15*	*20*
Doesn't like Bach	*45*	*20*

Let's examine the given information to see what it tells us.
The sum of all 4 cells must be 100.
The "Likes Bach" row must add up to 35.
The top left cell must be 15.
The bottom right cell must be 20.

So, we can subtract to find the other two cells.
Since 35 total people liked Bach, and of these 15 also liked Mozart, then 20 did not.
Of the 65 who did not like Bach, 20 also didn't like Mozart, so 45 did.

To answer the question, we need the sum of the Mozart column, which is 60.

51. A friend has given us a small cookbook. After perusing the ingredients of each recipe, we determine the following quantities:
    - n = total number of recipes
    - a = number of recipes that contain meat
    - b = number of recipes that contain onion
    - c = number of recipes that contain barbeque sauce
    - d = number of recipes that contain meat and onion
    - e = number of recipes that contain onion and barbeque sauce
    - f = number of recipes that contain meat and barbeque sauce
    - g = number of recipes that contain meat, onion, and barbeque sauce

    In terms of the variables, how many recipes do not contain any meat, onion, or barbeque sauce? (In other words, devoid of all three ingredients)

    This is an inclusion-exclusion problem.
    n(meat $\cup$ onion $\cup$ bbq)

    = n(meat) + n(onion) + n(bbq)
    – n(meat $\cap$ onion) – n(meat $\cap$ bbq) – n(onion $\cap$ bbq)
    + n(meat $\cap$ onion $\cap$ bbq)

    = a + b + c – d – e – f + g

    So, the number that do <u>not</u> contain any of these ingredients is:
    n – (a + b + c – d – e – f + g)
    = n – a – b – c + d + e + f – g.

52. A survey was taken of 360 students, to gauge the relative popularity of Latin, computer science, and psychology. It was found that, of these students, 260 were taking psychology, 170 were taking Latin, 130 were taking both psychology and computer science, and 90 were taking Latin but not computer science. Ten students were taking none of these three subjects. Therefore, how many students surveyed were taking computer science and Latin, but not psychology? If this number cannot be determined, what further information would you need?

    There are 360 students, and 260 are taking psychology. Therefore, 100 students are not taking psychology. We also know that 10 students are taking nothing. These 10 students are included in the 100 who were not taking psychology. Consider the other 100 – 10 = 90 students. What are they doing? They are not taking psychology, but they are taking either computer science or Latin, which is what the question asks. The answer is 90.

53. Suppose X = { 1, 2, 3, 4 } and Y = { 3, 4, 5 }. How many elements are in the set (X $\times$ Y) $\cap$ (Y $\times$ X) ?

    X $\times$ Y = all ordered pairs where x = 1,2,3,4 and y = 3,4,5
    Y $\times$ X = all ordered pairs where x = 3,4,5 and y = 1,2,3,4

    The intersection of these two sets is the set of ordered pairs where x = 3 or 4, and where y = 3 or 4. There are 4 such elements: { (3,3), (3,4), (4,3), (4,4) }.

54. Suppose x and y have the following 8-bit representations.

    x = 1010 1101

    y = 1111 0000

    What are the results of these bitwise operations? Assume they run independently of one another.

    a. x & y *1010 0000*

    b. x | y *1111 1101*

    c. x ^ y *0101 1101*

55. Suppose x is a bit vector (i.e. an integer with a binary representation). Explain what bitwise operation we should perform in order to invert the rightmost 3 bits of x, and leave the remaining bits of x unchanged.

    We need to perform an XOR operation with a mask whose rightmost 3 bits are 1, and remaining bits are 0.

56. Let n be an integer variable. Write an assignment statement using bitwise operators that will perform the following modification on the bits of n: the rightmost 2 bits should become zero, and the 3rd and 5th bits from the right end should become 1. The remaining bits should be unchanged.

    We need to "and" the rightmost 2 bits with 0, and "or" bits number 2 and 4 with 1.
    For the "and" operation, we need a mask whose binary representation has 0 in its rightmost two positions and 1 everywhere else: 11111…111100. In hexadecimal this is ~0x3.
    For the "or" operation, we need a mask whose binary representation has 1 in bits 2 and 4 and 0 everywhere else: 00000…0010100. In hexadecimal this is 0x14.

    Therefore, the desired assignment statement is:
    n = (n & ~0x3) | 0x14.

57. What is the overall effect of each of the following Java statements on the bits of x?

    a. x &= ~(1 << 8);

    Let's see how the mask is being calculated.

Start with 1:	*… 0000 0000 0000 0001*
Shift left 8:	*… 0000 0001 0000 0000*
Invert:	*… 1111 1110 1111 1111*

    The variable x is begin anded with this mask. Thus, almost every bit of x is being anded with a 1, which has no effect. The 9th bit from the right of x is anded with 0, so it is set to 0. So, the only thing that happens to x is its 9th bit from the right is cleared.

    b. x |= 0x1e;

    The number 0x1e in binary is 11110.
    This statement will modify x by turning on the 2nd, 3rd, 4th and 5th bit from the right end.

58. Consider the binary number 0000 1111 1100 0000, which may be used as a mask for some bitwise operation.
    a. The value of this binary number is the sum of consecutive powers of two, namely $2^x + 2^{x+1} + 2^{x+2} + \ldots + 2^y$. Therefore, x = _____ and y = _____.

       If we number the bits from 15 to 0, we see that bits 11 down to 6, inclusive are all 1's. The way the powers of 2 are written, y > x. So, x = 6 and y = 11.

    b. This summation can be simplified as a difference of powers of 2, namely $2^a - 2^b$. Therefore, a = _____ and b = _____.

       a = 12 and b = 6.

59. Suppose x is a 16-bit integer. Show how we can use a bitwise operation to set the leftmost 4 bits to 1, and leave the remaining 12 bits unchanged.

    To set to 1, we OR with a 1. ORing with a 0 has no effect. Therefore, to set the leftmost 4 bits, we need to OR x with:
    1111 0000 0000 0000

60. Let's warm up with some counting:
    a. Suppose in your foreign language class, you are studying for a vocabulary test. You need to study the forms of 10 verbs. For each verb you need to know how it is conjugated in the past, present, and future, and for first, second, and third person forms both in the singular and the plural. How many total words do you need to know?

       *Total = 10 * 3 * 3 * 2*

    b. Suppose you have designed an instruction cache simulator. It is set up to run any one of 30 different programs. When you run the simulator, the cache configuration can be set as follows. The set associativity can be any power of two from 1 to 16, inclusive. The number of lines in cache can be any power of two from 32 to 2^{20}. And the line size can be 4, 8, 16 or 32 bytes. How many different ways can your simulator be run?

       *Total = 30 * 5 * 16 * 4*

61. Suppose a social group consisting of five families of three, six families of four, and four families of five all go out to a ball game. In how many ways can they sit in a row of seats so that members of the same family all sit together?

    Permute the 15 families: 15!
    Each of 5 families of 3 has its members permuted: 3!
    Each of 6 families of 4 has its members permuted: 4!
    Each of 4 families of 5 has its members permuted: 5!
    Total answer = $15!\ 3!^5\ 4!^6\ 5!^4$.

62. In how many ways can 104 children be divided into four 16-member football teams, four 6-member hockey teams and four 4-member curling teams, such that teams playing the same game are indistinguishable? What if the teams are distinguishable?

    This can be thought of as a question of distinguishable permutations. You could also approach the problem using combinations, and see if you get an equivalent answer.

    We need to permute 104 people. But membership on a team means that the "position" within a team does not matter. So, in the denominator, we have a factorial for every team. We also note that teams of the same sport are indistinguishable. So, our answer is:
    $104! / (16!^4\ 6!^4\ 4!^4\ 4!^3)$

63. A CD contains 10 songs and it has a total running time of 44 minutes. How long would it take to play all possible permutations of the songs?

    *There are 10! ways to play the songs. Each permutation takes 44 minutes. So the total amount of time is 44*10! minutes. Incidentally, this works out to about 300 years.*

64. In how many ways can 2n men and 2n women pair up to play mixed doubles tennis? Assume that the players are only interested in who their partner is and who their two opponents are. They don't care which court or which end of the court they play on.

    This can be treated as a case of distinguishable permutations. We have to permute 2n men and 2n women. In other words, we have to arbitrarily assign 2n men to a certain side of a certain court, and do the same to the women. But note that each of the n tennis courts is considered indistinguishable. And on each court, the two sides are indistinguishable.
    So, our answer is: $(2n)!\ (2n)! / (n!\ (2!)^n)$

65. From a group of 20 married couples, how many ways are there to pick:
    a. a man and a woman who are not married to each other?

       *First choose the man. Then choose the woman who is not his wife. Answer: 20*19.*

    b. two people who are not married to each other?

       Let's solve this one indirectly. First, we calculate the total number of ways to pick any two people. Then subtract the cases where we choose a husband/wife pair.
       Answer: C(40, 2) – 20.

66. Consider the set of all 4-digit telephone numbers (0000-9999). How many such telephone numbers are there satisfying these criteria?
    a. All the digits are different.

       *This is the multiplication rule with no repetitions allowed. Answer: 10*9*8*7.*

    b. No two consecutive digits are the same.

*The first digit can be anything. But each remaining digit cannot match the one before it. So, we have 10*9*9*9.*

67. How many permutations of the letters ABCDEFG contain the substring "CAFE" ?

C A F E _ _ _	*There are four possible places where the substring could occur.*
_ C A F E _ _	*In case case, we have 3 remaining letters (B, D and G) that need to be*
_ _ C A F E _	*permuted in the other positions.*
_ _ _ C A F E	*Answer: 4(3!)*

68. How many 5-letter strings (from aaaaa to zzzzz) can be created, subject to the restriction that the letter 'i' may not appear immediately after a 'g'?

The gross number of strings is: $\underline{26}\ \underline{26}\ \underline{26}\ \underline{26}\ \underline{26} = 26^5$.

But we need to subtract those that contain the substring "gi".

$G\ I\ \underline{26}\ \underline{26}\ \underline{26}$
$\underline{26}\ G\ I\ \underline{26}\ \underline{26}$
$\underline{26}\ \underline{26}\ G\ I\ \underline{26}$
$\underline{26}\ \underline{26}\ \underline{26}\ G\ I = 4 * 26^3$.

Next, we need to add in cases that we took out twice.

$G\ I\ G\ I\ \underline{26}$
$G\ I\ \underline{26}\ G\ I$
$\underline{26}\ G\ I\ G\ I = 3 * 26^1$.

Our answer is $26^5 - 4 * 26^3 + 3 * 26^1$.

Note that if we had 6 or more letters in our string, we would need to continue this inclusion/exclusion.

69. Consider the set of all four-letter strings of capital letters from AAAA to ZZZZ.
    a. How many contain exactly one M?

M _ _ _	*Each other letter in our string is not an M.*
_ M _ _	*Answer =* $4\ (25)^3$.
_ _ M _	
_ _ _ M	

    b. How many contain at least one M?

Answer = (total) – (illegal)
= (total) – (having no M anywhere)
$= 26^4 - 25^4$.

70. An operating system requires users to have a password. Suppose the rules for selecting a password are as follows:

- The length is exactly 8 characters.
- Valid characters for a password are digits (0-9), capital letters (A-Z) and lowercase letters (a-z).
- The password must contain at least one digit.

Therefore, how many valid passwords exist?

Let's first find how many unrestricted passwords exist, then subtract the illegal ones that are missing a digit.
Number of unrestricted passwords: 62^8.
If we disallow digits: 52^8.
Total = $62^8 - 52^8$.

71. On a certain computer, an integer is internally stored as 32 bits. How many possible representations are there in which:

    a. exactly 5 of the bits are 1's?

    C(32, 5)

    b. at most 5 of the bits are 1's?

    C(32, 0) + C(32, 1) + C(32, 2) + C(32, 3) + C(32, 4) + C(32, 5)
    Summation notation would be handy in this problem.

    c. the first 5 bits are 1's?

    We have no choice about what to write for our first 5 bits, but the remaining 27 bits can be anything. Answer: 2^{27}.

    d. there is an equal number of 0's and 1's?

    This means we must choose which 16 positions are zero. Answer: C(32, 16)

    e. there are more 1's than 0's?

    The number of 1's is any number from 17 through 32, inclusive. So, our answer is the sum of all the terms of the form C(32, k) where $17 \le k \le 32$.

    f. exactly 5 of the bits are 1's, and the two leftmost bits are the same?

    There are two cases:
    First, if the first two bits are 1, then among the remaining 30 bits, we choose which 3 are 1.
    Second, if the first two bits are 0, then among the other 30 bits, we choose which 5 are 1.
    Answer: C(30, 3) + C(30, 5)

g. Let's look at some of our results. In part (a) the five 1's could be anywhere in the bit string. But in part (c) the five 1's had to be at the beginning of the bit string. Logically, can we say that we should expect the answer to part (a) to be larger than part (c)? Explain.

Be careful: the answer is no. There is a crucial difference between parts (a) and (c). In part (a), the other 27 bits had to be 0. In part (c), the remaining 27 bits could be either 0 or 1. Because of so much freedom, part (c) has a larger answer. In general, if we change the "five" to some other number, it could be that either part (a) or part (c) could have been larger. For example, if we change the 5 to 27, part (a) does not change, but part (c) is now just $2^5 = 32$, which is definitely less than C(32, 5). It may be interesting to find where the break-even point is.

72. Let's count some binary strings.

a. How many bit strings of length 10 either begin with three 0's, or end with two 0's, or both?

There are three types of binary strings we need to account for:
(bit strings starting with 000) + (bit strings ending with 00) – (double counting both)

If a binary string must start with 000, we may freely select the last 7 bits: 2^7.
If a binary string must end with 00, we may freely select the first 8 bits: 2^8.
If it must begin 000 and end 00, we may freely select 5 in the middle: 2^5.

Total = $2^7 + 2^8 - 2^5$.

b. How many bit strings of length 16 contain 14 or more consecutive zeros?

There are 3 possible places for these zeros to be located.

- *The first 14 positions*
- *The middle 14 positions*
- *The last 14 positions*

*In each of these 3 cases, we have the freedom to select the two other bits. Selecting values of two bits can be done in $2^2 = 4$ ways. So, there are 3*4 = 12 ways to write our binary number. But we need to watch out for duplicates whenever we select a 0 that is adjacent to the 14 consecutive zeros we started with:*

What if the 2 bits we select are 11? All three cases are distinct.
$11\ 0^{14}$
$1\ 0^{14}\ 1$
$0^{14}\ 11$

What if the 2 bits we select are 01? The second and third cases are identical.
$01\ 0^{14}$
$0\ 0^{14}\ 1$
$0^{14}\ 01$

What if the 2 bits we select are 10? The first and second cases are identical.
$10\ 0^{14}$
$1\ 0^{14}\ 0$
$0^{14}\ 10$

What if the 2 bits we select are 00? Then, all 3 strings are the same: 0^{16}.

As a result, there are 4 cases that we need to take away, leaving us with 12 – 4 = 8 strings.

c. How many bit strings of length 16 begin with 4 zeros and end with 4 ones?

We may freely select only the 8 bits in the middle. Answer = 2^8*.*

d. How many 16-bit integers have exactly 9 zeros, and begin and end with the same digit?

Consider the case where the first and last bits are 0. Then, among the 14 bits in the middle, 9 must be zero: C(14, 9).
Next, consider the case where the first and last bits are 1. Then, among the 14 bits in the middle, 7 must be zero: C(14, 7).
Add the two cases: C(14, 9) + C(14, 7).

73. The Leaning Tower Pizza Parlor makes just three kinds of pizza. Medium, Large, and Sicilian. In how many ways can someone order six pizzas?

The pizzas are identical except in what category we want them, so this is a ball-in-urn question. We have 6 objects and 2 separators. We need to decide which 2 of the 8 total positions are the separators. Answer: C(8, 2).

74. How many 5-card poker hands have:

a. a full house containing at least 1 ace?

We either have 2 aces or 3 aces.

Two aces means we have 3 of something else. Note: there are 12 other suits.
First, choose the other suit: C(12, 1)
Second, choose 3 of that suit: C(4, 3)
Third, choose the 2 aces: C(4, 2)

The three ace case can be done analogously
First, choose the other suit: C(12, 1)
Second, choose 2 of that suit: C(4, 2)
Third, choose the 3 aces: C(4, 3)

Interestingly, the two cases are numerically equal. Answer: 2 C(12, 1) C(4, 3) C(4, 2).

b. a full house containing 3 aces?

Well, we just want half of the previous answer! Answer: C(12, 1) C(4, 3) C(4, 2)

c. exactly four face cards (jack, queen, king)?

There are 12 face cards, and we want 4 of them. There are 40 non-face cards, and we want one. Answer: C(12, 4) C(40, 1).

d. as many diamonds as clubs?

There are 3 cases: either we have 2 of each, 1 of each, or 0 of each. The remaining cards in our poker hands will come from the other 26 cards in the deck. Answer: C(13, 2) C(13, 2) C(26, 1) + C(13, 1) C(13, 1) C(26, 3) + C(13, 0) C(13, 0) C(26, 5).

e. more diamonds than clubs?

Ugh, there are many cases. Handle each one, and add them all up. For sake of brevity I'll omit the C(n, r) notation factors that equal 1.

1 diamond, 0 clubs, 4 other	*C(13, 1) C(13, 4)*
2 diamonds, 0 clubs, 3 other	*C(13, 2) C(13, 3)*
3 diamonds, 0 clubs, 2 other	*C(13, 3) C(13, 2)*
4 diamonds, 0 clubs, 1 other	*C(13, 4) C(13, 1)*
5 diamonds, 0 clubs	*C(13, 5)*
2 diamonds, 1 club, 2 other	*C(13, 2) C(13, 1) C(13, 2)*
3 diamonds, 1 club, 1 other	*C(13, 3) C(13, 1) C(13, 1)*
4 diamonds, 1 club	*C(13, 4) C(13, 1)*
3 diamonds, 2 clubs	*C(13, 3) C(13, 2)*

f.cards from all four suits?

One of the suits will be distinguished by us taking two cards from it.
First, select which suit we want two cards from: C(4, 1).
Second, select two cards from that suit: C(13, 2)
Finally, select one card from each of the other suits: $C(13, 1)^3$

Multiply the above to write our total answer: $C(4, 1)\ C(13, 2)\ C(13, 1)^3$

g. at least 3 hearts?

This means 3, 4 or 5 hearts. Do each case separately and add them up. Note that there are 39 cards that are not hearts.
Answer: C(13, 3) C(39, 2) + C(13, 4) C(39, 1) + C(13, 5) C(39, 0).

h. exactly 2 clubs and exactly 2 kings?

Split this up into two cases depending on whether we have the king of clubs or not.

Yes case: We want the king of clubs, 1 more club that is not the king, and 1 more king that is not a club. The other 2 cards are selected from the rest of the deck. How many cards are neither king nor club? That is 52 – 4 – 13 + 1 = 36. So, the number of ways here is C(1, 1) C(12, 1) C(3, 1) C(36, 2).

No case: We want two clubs but neither one being a king, two kings but neither one being a club, and then 1 of the other 36 cards. This is C(12, 2) C(3, 2) C(36, 1).

Total answer: C(1, 1) C(12, 1) C(3, 1) C(36, 2) + C(12, 2) C(3, 2) C(36, 1).

i. the same number of diamonds, hearts and clubs?

We either have 1 of each or 0 of each. Therefore, the number of spades in our hand is either 2 or 5, respectively. Just remember than in both cases, we want a total of 5 cards.
Answer: $C(13, 1)^3\, C(13, 2) + C(13, 0)^3\, C(13, 5)$

j. no kings?

If we can't have a king, then we must select our 5 cards from among the non-kings in the deck. Our answer is C(48, 5).

k. exactly one king?

We choose one of the kings, and then the other 4 cards are non-kings.
Answer: C(4, 1) C(48, 4).

l. at least one king?

"At least one" is the opposite of having none. Answer: C(52, 5) – C(48, 5).
You could also have approached this problem by enumerating all of the possible cases: one king, two kings, three kings, and four kings, and adding these answers. It's a little more tedious, but here it is:

1 king: C(4, 1) C(48, 4)
2 kings: C(4, 2) C(48, 3)
3 kings: C(4, 3) C(48, 2)
4 kings: C(4, 4) C(48, 1)
Total answer is the sum of these four numbers.

m. at least two kings?

In this case, at least 2 means 2 or 3 or 4. We have already worked out these cases in the previous part, so our answer is: C(4, 2) C(48, 3) + C(4, 3) C(48, 2) + C(4, 4) C(48, 1).

Once again, you could have worked this one indirectly. "At least two" is the opposite of "at most one," and to solve the at-most-one, there are only two cases to consider: the no-king case and the one-king case. Our answer would be: C(52, 5) – C(4, 0) C(48, 5) – C(4, 1) C(48, 4).

n. exactly two kings, exactly two queens, or both?

Here, we need to avoid double counting.
 # with two kings + # with two queens – # number with both
= C(4, 2) C(48, 3) + C(4, 2) C(48, 3) – C(4, 2) C(4, 2) C(44, 1)

o. Which of the above combinations of answers should sum to C(52, 5)?

Parts (j) and (l) should add up to C(52, 5), and parts (j), (k) and (m) should also add up to C(52, 5). The reason why is that these combinations of cases encompass all possible poker hands.

75. Saturn poker is a game much like regular poker, except that the hands have six cards instead of five. In Saturn poker, it is possible to be dealt three pairs. For example, a pair of aces, a pair of kings, and a pair of queens. How many Saturn poker hands contain three pairs?

First, choose the three denominations: C(13, 3).
Then, choose two from each denomination: C(4, 2) three times.
Total: C(13, 3) $C(4, 2)^3$.

76. To play a lottery, you select six different numbers between 1 and 50, inclusive. A lottery drawing consists of six distinct winning numbers. The grand prize is awarded to the player who selected all six winning numbers. But smaller prizes are available to players who selected only some of the winning numbers. How many ways are there to select six numbers from 1 to 50, inclusive, if exactly four of the six are winning numbers?

We want 4 of the 6 winning numbers and 2 of the 44 losing numbers.
Answer: C(6, 4) C(44, 2).

77. Find the probability of each of the following events when rolling five dice.

a. Five-of-a-kind: All dice show the same number.

Throughout this problem, the denominator, the number of total outcomes of rolling the dice, is 6^5.

*There are 6 ways getting all the numbers to match. (The first die can be anything, but then all the other dice have only 1 choice.) And then if we try to permute the order of the dice, we see that it doesn't really matter. Once we have selected the numbers we want on the dice, the number of ways to permute their order is 5!/5!, which is 1. Our numerator is 6*1, so the probability is $6/6^5$.*

When I'm not sure my intuition is right on solving a problem, I try to solve it a different way. Or in this case I can write a program to generate all possible cases to compute the probability, or run random cases to estimate the empirical probability.

I wrote a program to generate all 7776 cases, and it told me that there are 6 five of a kinds, 150 four of a kinds, 300 full houses and 240 straights.

b. Four-of-a-kind: Exactly 4 of the 5 dice show the same number.

*Let's first select what numbers we want. The first die can be anything, but dice 2-4 must match. The fifth die can be anything other than what the first die is. So, the number of possible ways to get 4 of a kind in this order is 6*1*1*1*5 = 30. But these dice need to be* *permuted in various orders. Keep in mind that 4 of the 5 dice are the same number. The number of distinguishable permutations is 5!/4!= 5. As a result, the total number of ways to get a 4 of a kind is 30*5 = 150.*

c. A full house: three dice show the same number, and the other two match some other number.

*Let's first select the numbers we want. The first die can be anything, but the next two must match it. Then the 4^{th} die can be anything other than what the first die is. And the last die has to match the 4^{th} die. So, the number of ways to do this is 6*1*1*5*1 = 30. Now we have to permute the order of these dice. There are 5 dice, but 3 are the same, and 2 others are the same, so the number of permutations is 5!/(3! 2!) = 10. The total number of ways to have a full house, therefore, is 30 * 10 = 300.*

d. Straight: The dice show the numbers 1-5 or 2-6.

*Let's first select the numbers. The first die is either a 1 or a 2. This choice automatically determines what the remaining numbers are. So the number of ways to select the numbers is 2*1*1*1*1 = 2. Next, we permute these numbers. They are all different, so there are 5! = 120 permutations. As in the other cases, we multiply to obtain our final answer: 2*120 = 240.*

e. What is the most likely sum of the dice? What is the probability of achieving this sum?

We can solve this by generating functions. We are looking for the largest coefficient of $(x + x^2 + x^3 + x^4 + x^5 + x^6)^5$. From Wolfram Alpha, we see that this generating function works out to $x^5 + 5x^6 + 15x^7 + 35x^8 + 70x^9 + 126x^{10} + 205x^{11} + 305x^{12} + 420x^{13} + 540x^{14} + 651x^{15} + 735x^{16} + 780x^{17} + 780x^{18} + 735x^{19} + \dots + x^{30}$. The largest coefficient occurs for a sum of dice of 17 or 18. This is not surprising because these numbers are in the middle of the list from 5 to 30. The probability of either case is $780/6^5$, which is slightly more than 10%.

78. Registration PINs consist of 4 characters: a letter, a digit, and then 2 more letters. If each Furman student receives 3 different PINs during the year, and the registrar attempts to give unique PINs every term, how long will it take for all possible PINs to be used up?

*The number of PINs available is 26*10*26*26. This works out to 175,760. The student population at Furman is about 2650. So the number of years is 175760/(3*2650) = 22.*

79. In how many ways can we divide twenty people into teams of sizes 4, 5, 5 and 6?

We are permuting the positions of 20 people, but the positions within each team are indistinguishable. And the two teams of 5 are also indistinguishable.

Answer: 20! / (4! 5! 5! 6! 2!)

80. How many distinct strings can be formed in each of the following situations?

a. Using all the letters of WORKING, but the letter K is not in the middle position

All the letters are distinct, so the total number of ways to permute the letters is 7! How many ways put the K in the middle? This would leave us freedom to place only 6 letters, and this case would be 6! Answer: 7! – 6!

b. Using all the letters of FEELING, but the letter E is not in the middle position

*First of all, let's figure out how many ways this word can permute its letters. It's a 7-letter word, and the only repetition is the pair of E's. So, the number of distinguishable permutations is 7!/2!. But now, we must exclude all of the permutations where an E is in the middle position. Obviously, some letter must occupy the middle position. What is the probability that it is an E? There are 2 E's, so this probability is 2/7. Thus, there is a 5/7 chance that a given permutation is acceptable. Thus, our final answer is (5/7) * 7!/2!.*

c. Using five of the letters taken from ELEMENT

In this problem, our answer will depend on how many E's we select. So, let's split up into various cases, and add up all our answers.

Select no E. This is impossible because we must make a 5 letter word, and only 4 letters are not E. So, this case yields 0.

Select one E. Now we have 5 unique letters, so there are 5! ways to arrange them.

*Select two E's. In this case, we must select 3 of the 4 letters other than E, and this can be done in C(4, 3) ways. Then we have a 5-letter word with a double E. The number of distinguishable permutations is 5!/2!. So, this case yields C(4, 3) * 5!/2!*

*Select all 3 E's. Now, we must select 2 of the 4 letters other than E, which can be done in C(4, 2) ways. Then we have a 5-letter word with 3 E's. The number of distinguishable permutations here is 5!/3!. So, this case yields C(4, 2) * 5!/3!*

Our final answer is 5! + C(4, 3) 5!/2! + C(4, 2) 5!/3!.

81. Getting in line…

a. A bookshelf contains 4 physics books, 5 chemistry books, 3 history books and 6 economics books. In how many ways can these books be arranged on the shelf so that books on the same subject are grouped together, and the science books are grouped together as well?

Permute within each subject: 4! 5! 3! 6!
Permute the two science subjects: 2!
Permute all subjects, but keeping science combined as one unit: 3!
Multiply everything to obtain final answer: 4! 5! 3! 6! 2! 3!

b. In how many ways can 5 boys and 5 girls get on a lunch line at school in such a way that the boys and girls alternate positions? If there were 6 boys and 5 girls, how many ways can they line up alternating?

If the boys and girls alternate positions, we need to decide if the boys or the girls come first. Then we would permute the boys and girls separately. Answer: 2 5! 5!.*

c. How many ways can 8 men and 5 women seat themselves in a line of chairs so that no two women sit together? (In other words, between every two women, there must be at least one man.)

First, we need to find the number of valid configurations of where men and women may sit among each other. Picture this: ___ W ___ W ___ W ___ W ___ W ___. We are required to place 4 men between the first and last women. The other 4 men can be placed in any one of the 6 places among the women. In other words, we have 4 identical objects to place into 6 categories. We have 4 objects and 5 separators, so the number of configurations is C(9, 4).

Next, we need to permute the men and women separately. This can be done in 8! 5! ways. Our total answer is C(9, 4) 8! 5!

82. A bookshelf contains 7 books written in German, 6 books written in French, and 5 books written in Swedish. In how many ways can the books be arranged on the shelf if …

a. The books are all distinct, and books of the same language are grouped together?

Permute books of each language. Don't forget to also permute the languages.
7! 6! 5! 3!

b. The books do not need to be grouped by language, but books of the same language are identical?

Distinguishable permutations. 18! / (7! 6! 5!)

83. Fill in the blanks.

a. There are ___*n!*___ ways to arrange n books on a bookshelf.

b. There are ___*(n – 1)!*___ ways to arrange n people around a dining room table.

c. There are ___*(n – 1)! / 2*___ ways to arrange n keys around a key chain.

d. The number of ways to select 5 objects from a set of 20 is equal to the number of ways to select ___*15*___ objects from the same set.

84. How many 9-digit numbers contain three 7s, three 8s, and three 9s?

Choose which 3 digits are 7s, which can be done in C(9, 3) ways. Then, choose positions for the 8s, which is C(6, 3). Finally, the 9s really have no choice because there are only 3 places left, C(3, 3). Total answer is C(9, 3) C(6, 3) C(3, 3), and you could omit the last factor because it equals 1.

You could also approach this problem as distinguishable permutations. Let's say we have the number 777888999, and we want to know the number of ways to permute the digits. It would be: 9! / (3! 3! 3!). Note that the digits 7, 8, and 9 are distinct, so we do not need to divide by another factorial.

85. Suppose a set has ten elements. How many ways are there to select a subset of elements, under each of the following scenarios?
    a. There are no restrictions on the size or membership in the subset.

    There are 2^{10} possible subsets of a set containing 10 elements.

    b. The subset must contain exactly 4 elements.

    We must select 4 out of 10. This is a classic combination question. Answer: C(10, 4).

    c. The subset must contain exactly 7 elements, and element "A" must be included in the subset.

    We must select the A, which can be done in C(1, 1) ways. For the remaining 9 elements, we are free to select any 6 of them, which can be done in C(9, 6) ways. To obtain our total answer, we multiply, but note that C(1, 1) = 1. So, the answer is just C(9, 6).

86. Suppose S is a set and S = { 2, 4, 6, 8, 25, 50, 75, 100, 200, 300, 400 }. How many subsets of S contain exactly two 1-digit numbers, exactly two 2-digit numbers, and any quantity of 3-digit numbers?

    Choose 2 from (2, 4, 6, 8), choose 3 from (25, 50, 75), and choose any of the last 4.
    Answer: C(4, 2) C(3, 2) 2^4.

87. Determine whether the following statement is true or false, and justify your answer. For all positive integers a, b, and c,

$$\binom{a+b+c}{a}\binom{b+c}{b} = \binom{a+b+c}{c}\binom{a+b}{a}$$

Let's simplify by working out the factorial definition of C(n, r).

$$\frac{(a+b+c)!}{a!\ (b+c)!}\ \frac{(b+c)!}{b!\ c!} = \frac{(a+b+c)!}{c!\ (a+b)!}\ \frac{(a+b)!}{a!\ \ b!}$$

We see that both sides of the equation have (a + b + c)! in the numerator and a!b!c! in the denominator. Eliminating these, we have:

$$\frac{(b+c)!}{(b+c)!} = \frac{(b+c)!}{(b+c)!}$$

which simplifies to 1 = 1, which is true.

Incidentally, you could give a combinatorical argument. The left side of the equation models the situation of having a+b+c people and first selecting a of them. Of the remaining b+c people, selecting b of them, leaving c to choose by default.

The right side of the equation says we have a+b+c people, and we select c of them first. Of the remaining a+b people, we select a of them, leaving b to choose by default.

In effect, each side of the equation is dividing a group of a+b+c people into groups of size a, b, and c, assuming that these numbers are distinct (or there is some other property that makes the three groups distinct). The only difference is the order in which the teams are selected, which does not matter. We would obtain the same numerical result, so the equation holds.

88. Let's flip some coins.
   a. Suppose you flip a coin seven times. Each flip of the coin is considered distinct. In how many ways is it possible to get at least three heads and at least two tails?

      There are three possible cases: we have 3 heads, 4 heads or 5 heads.
      Answer: C(7, 3) + C(7, 4) + C(7,5).

   b. A coin is flipped 7 times. What is the probability that exactly 4 outcomes are heads and 3 are tails?

      Choose which outcomes are heads. Or equivalently, choose which ones are tails. When I write my combination expressions C(n, r), I generally prefer r to be no more than n/2, but this is just a personal preference. So, the number of favorable outcomes is C(7, 3). The total number of outcomes is 2^7. *Thus, the probability is C(7, 3) /* 2^7.

      Incidentally, this probability works out to be 35 / 128, and this is tied for the most likely outcome of flipping a coin 7 times.

   c. If you flip a coin 10 times, what is the probability that you will see heads come up at least once?

      P(one tail) = 1/2
      P(ten tails) = $(1/2)^{10}$
      P(at least one head) = 1 – P(ten tails) = 1 – $(1/2)^{10}$

89. It's time for a committee meeting!
   a. A social club consists of 10 men and 20 women. A finance committee of 5 individuals needs to be formed. How many ways can this be done if at least 2 members of the committee must be women?

      This means we have 2, 3, 4 or 5 women. Do each case, and add up our answers.
      Two women: C(10, 3) C(20, 2)
      Three women: C(10, 2) C(20, 3)
      Four women: C(10, 1) C(20, 4)
      Five women: C(10, 0) C(20, 5)

      Alternatively, you could have calculated the number of ways of choosing 0 or 1 woman, and then subtract from all possible 5-member committees: This way, our answer would be C(30, 5) – C(10, 5) C(20, 0) – C(10, 4) C(20, 1)

b. A company employs 28 men and 30 women. A committee of at least 6 but no more than 10 employees needs to be formed. In how many ways can this be done if the number of men and women on the committee must be the same?

We need to have a committee of even size. That is, size 6, 8 or 10.
6 people (3 men & 3 women): C(28,3) C(30,3)
8 people (4 men & 4 women): C(28,4) C(30,4)
10 people (5 men & 5 women): C(28,5) C(30,5)

Since the three cases are mutually exclusive, we can add them up and not worry about subtracting any overlap.
Answer: C(28,3) C(30, 3) + C(28,4) C(30,4) + C(28,5) C(30,5)

90. Athens and Sparta agree to hold a peace conference to avert a war. Each side will send four delegates. Athens will select four of its ten generals. Sparta will send four of its 28-member Gerousia. However, the Spartan side is rather intimidated by one Athenian general in particular, Pericles. Sparta insists that if Athens selects Pericles as one of their representatives, then Sparta will send five delegates instead of four to the conference. Therefore, in how many ways can the total membership at the peace conference be selected?

There are two mutually exclusive cases, so our answer is their sum.
Case I: Athens selects Pericles and three other generals. Sparta selects five. This part equals C(1, 1) C(9, 3) C(28, 5).
Case II: Athens does not select Pericles. Sparta selects four. This part equals C(1, 0) C(9, 4) C(28, 4).
Total: C(1, 1) C(9, 3) C(28, 5) + C(1, 0) C(9, 4) C(28, 4).

91. Suppose you run a theater company, and you need to plan on which days to schedule performances. For the month of April there are 22 weekdays (Mon-Fri) and 8 weekend days (Sat/Sun). How many ways can you choose exactly 10 of these days to perform given that at least 6 of these performance days must be weekend days?

We have the following cases: 6, 7 or 8 of the performances are on weekend days.
4 weekday and 6 weekend: C(22, 4) C(8, 6)
3 weekday and 7 weekend: C(22, 3) C(8, 7)
2 weekday and 8 weekend: C(22, 2) C(8, 8)
Add up the above three numbers.

92. How many different images exist? How much storage space would be needed to store them all? Assume that each image can be accommodated on your computer screen. What other assumptions do you need to make?

Let's make the following reasonable assumptions.
- *The image, which can fit on the screen, has 1 million pixels.*
- *Each pixel of the image can be one of 256 different colors.*

Therefore, the number of images would be 256 to the power of a million. If each image is 1 MB, we need $256^{1000000}$ MB of storage space. Talk about science fiction!

Incidentally, if we wanted to store images of any size up to 1 million pixels, we could invent a 257^th^ color representing a nonexistent or unused pixel in the image.

We can reduce our storage needs by enforcing a couple of restrictions. We can:

- *use smaller images, say just 50,000 pixels.*
- *reduce the color palette to just 16 colors.*
- *assume that compression allows us to reduce the average amount of space per pixel by half.*

*Now, each pixel occupies just 3 bits, and we need 50,000 pixels. Thus, we need only 19 KB for each image. But how many images are there? This is still the number of colors raised to the power of the number of pixels. This is 16^{50000}. The storage space needed is 19 * 16^{50000} KB. The number of kilobytes as a power to 10 is about 10^{60207} KB. What is this in TB instead? Unfortunately, converting from KB to TB only reduces the exponent on 10 by 9. So we would need 10^{60198} TB. The average computer today has a disk space of on the order of 1 TB. How many computers are in the world? To account for large servers, a generous estimate would be 1000 computers for each person in the world. So, let's say 10^{13} computers. Therefore, the total disk space on all computers in the world could be very roughly 10^{13} TB. But we need 10^{60198}. The computers of how many planet Earths would this be? We could divide and obtain: 10^{60185} planets. The number of planets in the universe is likely less than 10^{25}. Even the number of atoms in the universe is well under a googol. So, the entire universe is not large enough to store even a tiny fraction of all the possible images. Plus, it wouldn't be much good if an image you wanted happened to be billions of light years away.*

Don't worry about the exact orders of magnitude because it depends greatly on our assumptions. The bottom line is that you aren't going to be able to store all possible images on your computer!

93. For this problem write both exact answers and decimal approximations. A lottery drawing consists of six different numbers in the range 1-50. What is the probability of winning the lottery if you play with:
    a. One ticket?

       There are C(50, 6) ways in which the winning lottery numbers can be picked. So, the probability of winning the lottery is 1 / C(50, 6).

    b. One thousand tickets?

       Let's assume that when you bought the 1000 tickets, they were all different! In this case, the probability of winning is 1000 / C(50, 6). If any of your tickets are the same, then your chances of winning would be based on the number of distinct tickets.

94. A bag contains 20 balls: 8 blue, 6 red and 6 green. Assume that balls of the same color are identical.
    a. How many ways can we select 5 balls from the bag?

       This is a ball in urn problem. We have 5 identical objects and 3 categories. Hence, 2 separators. Total possibilities: C(7, 2).

    b. How many ways can we select 5 or fewer balls from the bag?

       We can add a new category called "unchosen ball." Our five balls now have 4 categories. Hence, 3 separators. Total possibilities: C(8, 3).

c. How many ways can we select 10 balls from the bag?

The ball in urn formula in general is C(n + r – 1, r – 1). We let n = the number of balls and r = the number of urns, categories or colors. If we plug in n = 10 and r = 3 we obtain C(12, 2).

But not so fast! It turns out that some combinations are impossible, and they need to be subtracted out of our answer. We could be specifying too many blue, or too many red, or too many green. What does it mean to have too many blue? It would mean insisting on at least 9 blue balls be chosen. Then we have freedom to select 1 more ball. We can use the ball-in-urn formula for n = 1 and r = 3, to obtain C(3, 2).

Next, we look at too many red. This would mean that we have at least 7 red balls. So, 3 more balls can be freely chosen. Use n = 3 and r = 3 in the ball-in-urn formula: C(5, 2). The case for green is the same.

Our final answer is C(12, 2) – C(3, 2) – 2 C(5, 2).

d. If we select 5 balls from the bag, what is the probability that we will get no green balls? (Assume that balls of the same color are identical.)

Numerator (favorable outcomes): Pretend that the green category doesn't exist. So, use the ball-in-urn formula for n = 5 and r = 2.
Denominator (total outcomes): Same answer as part (a), the ball-in-urn formula for n = 5 and r = 3.

Answer: C(6, 1) / C(7, 2).

95. Six divers in the Florida Keys discover the ruins of a Spanish galleon on the sea floor. Among the ruins are 100 gold coins. How many ways can these coins be distributed among the divers assuming that the coins are identical and each diver will receive at least 10 coins?

We only have complete freedom to distribute 40 gold coins among the six divers. Model the scenario as 40 identical objects with 6 categories or 5 dividers, and we obtain a 45-bit string with 40 zeros and 5 ones. We just have to decide where the 1s go. Answer = C(45, 5).

96. Suppose you have a bucket of 50 fish and your job is to feed 10 hungry dolphins. If you randomly toss fish into the water, what is the probability that all the dolphins get at least one fish?

Numerator: We are obliged to give each dolphin one fish, so we only have the freedom to distribute 40 fish. Ball-in-urn with n = 40 and r = 10.
Denominator: Ball-in-urn with n = 50 and r = 10.
Answer: C(49, 9) / C(59, 9).

97. A bagel shop sells 5 kinds of bagels. You want to buy at least 1 of each type of bagel, and you have enough money to buy 20 bagels. Therefore, in how many ways can you place an order of bagels that you can afford?

We must buy 5 bagels. So, we can freely choose how to allocate the other 15. Since we can choose to buy any number of additional of bagels from 0 up to 15, we can let the unbought bagels be the 6th type of bagel. Now, we have a ball-in-urn problem with 15 balls and 6 urns. Six urns means 5 separators. Think of a binary string with 15 zeros and 5 ones. Answer = C(20, 5).

98. A cookie jar contains 40 cookies: 10 chocolate chip, 10 oatmeal, 10 linzer, and 10 peanut butter. Cookies of the same type are indistinguishable. You want to select 8 cookies to eat. In how many ways can you make the selection if at least 2 cookies must be oatmeal?

    4 categories → 3 separators
    There are enough cookies of each type to avoid running out.
    8 – 2 = 6 cookies may be freely selected.
    So, we have 6 identical objects, and 3 separators. Answer: C(9, 3).

99. A bowl contains an unlimited supply of each of 21 varieties of Halloween candy. You want to take ten pieces of candy. How many ways could you make the selection if …
    a. The order in which you take the candy does <u>not</u> matter?

       Ball-in-urn problem: We have 10 identical objects and 21 categories. We can place 20 dividers between the categories. This is similar to asking how many binary numbers have 10 zeros and 20 ones. The answer is C(30, 10).

    b. The order in which you take the candy <u>does</u> matter?

       *For each piece of candy, we can choose any one of 21 varieties. This is the multiplication rule, so our answer is 21 * 21 * 21 … for all ten choices. Answer is 21^{10}.*

    c. The order in which you take the candy does not matter, and two of the 21 varieties each have only six pieces available (the other 19 varieties being unlimited)?

       We start with our answer to part a, and subtract the impossible cases. Either of two varieties must have at least 7 pieces of candy. This means we may freely choose 3 pieces of candy, still with 21 categories. For one of our restrictions, we have 3 identical objects and 21 categories. This means 20 dividers, and the number of combinations is C(23, 3). But since there are two varieties that have this restriction, be sure to count it twice. Also note that we won't have both restrictions at the same time because we are only selecting 10 candies. If we were selecting 12 or more (6+6) then we would have to take into account the possibility of double counting the restriction.

       Answer = C(30, 10) – 2 C(23, 3).

100. Suppose you have two dice. To roll "snake eyes" means to roll double ones. If you roll the pair of dice 100 times, what is the probability that you will roll snake eyes at least once?

    P(at least once out of 100) = 1 – P(never out of 100)
    = 1 – P(not rolling snake eyes)^{100}
    = 1 – (35/36)^{100}

101. Suppose you are an Olympic athlete, and at the Olympics there are 2800 athletes, and 500 medals will be given out.

a. What is the probability that you will receive a medal if the medals are awarded at random, and no one is allowed to receive more than one?

500/2800

b. Re-do part (a), but assume that there is no restriction on the number of medals one may receive.

If the medals are truly awarded at random, this means that each time a medal is awarded, any one of the 2800 athletes could win it. Each awarding of a medal is an event. For this event, you could win the medal or not.

$$
\begin{aligned}
P(\text{win at least 1}) &= 1 - P(\text{win none}) \\
&= 1 - P(\text{not win } 1^{st} \text{ and not win } 2^{nd} \text{ and … and not win } 500^{th} \text{ medal}) \\
&= 1 - P(\text{not win 1 medal})^{500} \\
&= 1 - (1 - 1/2800)^{500}
\end{aligned}
$$

c. Use a calculator to approximate your answers to parts (a) and (b) to the nearest thousandth. Explain why your answers make sense.

0.179 and 0.164
It should be more difficult for you to earn a medal if other people can take more than one. In other words, fewer competitors will receive at least one medal.

d. If no athlete may receive more than one medal, then clearly the number of athletes who receive a medal is 500. But if we remove this restriction, what is the expected number of athletes who receive at least one medal?

Just multiply the answer to part (b) by 2800. The result is 458.

102. A university has 30 computer labs, each with 25 computers. Let p be the probability that a given machine is not functioning.

a. What is the probability that the university has at least one computer lab where all its machines are not functioning?

p = probability of 1 machine not working
p^{25} = probability that all 25 machines in a lab don't work
$1 - p^{25}$ = probability that at least 1 machine works in a lab
$(1 - p^{25})^{30}$ = probability that at least 1 machine works in all 30 labs

Finally, we want the opposite of that last answer, which is:
$1 - (1 - p^{25})^{30}$

b. What is the probability that all of the computer labs have at least one computer that is not functioning?

p = probability of 1 machine not working

$1 - p$ = probability of 1 machine working
$(1 - p)^{25}$ = probability of all 25 machines in a lab working
$1 - (1 - p)^{25}$ = probability that at least 1 machine in one lab doesn't work

We want the above to take place in all 30 labs, so our answer is:
$(1 - (1 - p)^{25})^{30}$

c. Let p = 0.2. Use a calculator to evaluate the answers you found in parts (a) and (b). Explain why we should have expected one answer to be much larger than the other.

$a = 1.0066 * 10^{-16}$
$b = 0.8927$

Does your intuition agree that scenario B is much more likely than scenario A? This is interesting because both scenarios are satisfied by having a similar number of computers fail. But scenario A is more restrictive because it requires that the failing machines all be in a certain place, while scenario B doesn't care how the bad machines are distributed across the university.

Scenario A is about as restrictive as saying that we always want a specific computer, e.g. the one nearest the front door, to be a bad one in every lab.

d. Re-work parts (a) and (b), this time assuming that we have n computer labs and n computers in each lab. To what values do your two probabilities approach as n approaches infinity?

Part a: $p = 1 - (1 - p^n)^n$, which tends to 0 as n goes to infinity.
Part b: $p = (1 - (1 - p)^n)^n$, which tends to 1 as n goes to infinity.

103. Suppose that the probability of an event occurring once is 1/n, and there are n trials. We are interested in computing the probability that the event will occur at least once over the n trials.

a. What is the formula for this probability in terms of n?

$1 - (1 - 1/n)^n$

b. Use a calculator to determine this probability (to the nearest millionth) if n = 1 million.

0.632121

c. Use Wolfram Alpha to help you determine the probability when n is arbitrarily large. Your answer will make use of a famous mathematical constant.

$(e - 1) / e$

104. Use combinations to determine the value of count after the following code executes.

```
count = 0;
for (a = 0; a <= 100; ++a)
  for (b = 0; b <= 200; ++b)
    for (c = 0; c <= 300; ++c)
       for (d = 0; d <= 400; ++d)
          if (a + b + c + d == 100)
```

```
++count;
```

We have 100 identical objects that need to be put into 4 distinct categories. Using the ball-in-urn approach, we note that we would need 3 separators along with our 100 "coins," and we'd have to decide which 3 of the 103 positions are where the separators would go. So, our answer is C(103, 3).

105. The Paladin Express is a train designed to convey both horses and people. It consists of 26 cars: an engine, 5 horse cars and 20 passenger cars. All the cars are distinct. How many ways can the order of the cars be arranged so that **both** of these conditions are satisfied?
- The engine must be at the front of the train, and one of the passenger cars must be at the back.
- The horse cars must be spread out as follows: between each horse car and the next, there must be at least three passenger cars.

This is both a ball-in-urn problem and a permutation problem. Use the ball-in-urn idea to determine where the horse cars should go in relation to the passenger cars. Then we will permute the cars. The train is organized as follows, where E is the engine, H is a horse car, and P is a passenger car:

E _____ H _____ H _____ H _____ H _____ H _____ P

Notice the 6 blank areas of the train. This is where we must place 19 additional passenger cars. Because of our second constraint, the middle 4 blanks must each contain at least 3 passenger cars. So, we can update our diagram:

E _____ H PPP _____ H PPP _____ H PPP _____ H PPP _____ H ____ P

Now, we have freedom to place only 20 – 1 – 12 = 7 passenger cars. Seven identical objects among six distinct categories (five dividers). We can model this ball-in-urn problem as a binary number with seven 0s and five 1s. This can be done in C(12, 5) ways.

Next, we permute the horse cars and passenger cars separately: 5! 20!
Our final answer is C(12, 5) 5! 20!

106. In Pascal's triangle,

a. What is the fourth number in the tenth row?

*The rows and "columns" of Pascal's triangle are numbered from zero. So, the question is asking for C(9, 3), which equals (9*8*7)/(3*2*1) = 3*4*7 = 84.*

b. Does the number 13 appear anywhere? Explain.

Yes. As you read down the triangle, looking only at the second number on each row (or, alternatively, the penultimate number), you are reading the sequence of positive integers. The reason why is because C(n, 1) = n. Thus, we eventually reach the number 13, or any positive integer. The second number on the 14^{th} row is 13.

c. The number 1 appears an infinite number of times. Does any other positive integer appear an infinite number of times in Pascal's triangle? Prove your answer.

The answer is no.

Suppose n is a positive integer greater than 1. Where can this number reside in Pascal's triangle? It appears twice in row n. (Well, there is the special case of 2 appearing once.) Row n of Pascal's triangle begins with the numbers 1 and n, followed by larger numbers, and finishing with the last two values n and 1. It's also possible for n to appear in earlier (shorter) rows of the triangle. But let's see why it cannot appear on any deeper (longer) row.

Consider row n+1. The numbers on this row are 1, n + 1, some larger numbers, and then finally n + 1 and 1. The number n does not appear on this row because its value is strictly between 1 and n+1. In other words n is strictly between the first two or last two numbers on row n+1.

The situation is the same for any row t where t > n. The numbers on this row are 1, t, larger numbers, and then t and 1. The number n is not on this row because 1 < n < t. We skipped the number n.

Therefore, the number n cannot appear on any row deeper than row n. Thus, all of the occurrences of n appear in the first n+1 rows of Pascal's triangle. There are a finite number of values in these rows, so n appears only a finite number of times.

d. Let PT(r, c) be the number in row r and column c of Pascal's triangle. What is the recursive rule that defines PT(r, c)? In other words, express PT(r, c) in terms of smaller numerical parameters.

PT(r, c) = PT(r – 1, c – 1) + PT(r – 1, c)
Also note we have base cases:
PT(r, c) = 0 if r < 0 or c < 0 or c > r
PT(r, c) = 1 if c = 0 or c = r

e. Suppose a row begins with the numbers 1, 25, and 300. What are the first three numbers on the next row?

The first number of every row of Pascal's triangle is 1. In general, the rule for interior numbers is to add the two numbers above. Therefore, the first number is 1, the second number is 1 + 25 = 26, and the third number is 25 + 300 = 325.
Answer: 1, 26, 325.

107. Consider this list of eight numbers: L = (1, 1, 2, 2, 3, 3, 4, 4). We wish to know how many ways there are to make a selection of numbers from L that has a sum of 10. Show how to solve this problem using the generating function technique.

Each distinct number in the list gets its own factor in the generating function. The only thing we need to keep track of is the sum of the numbers in a selection. This will be the exponent on x.

We may select zero, one or two of the 1's: $(1 + x + x^2)$

Similarly, we may select 0/1/2 of the 2's: $(1 + x^2 + x^4)$
We may select 0/1/2 of the 3's: $(1 + x^3 + x^6)$
We may select 0/1/2 of the 4's: $(1 + x^4 + x^8)$

Therefore, our answer is the coefficient of x^{10} *in the generating function* $(1 + x + x^2)(1 + x^2 + x^4)(1 + x^3 + x^6)(1 + x^4 + x^8)$. *According to Wolfram Alpha, the expanded form of this generating function is* $1 + x + 2x^2 + 2x^3 + 4x^4 + 4x^5 + 5x^6 + 5x^7 + 7x^8 + 6x^9 + 7x^{10} + 6x^{11} + 7x^{12} + 5x^{13} + 5x^{14} + 4x^{15} + 4x^{16} + 2x^{17} + 2x^{18} + x^{19} + x^{20}$. *Our desired coefficient is 7.*

108. Consider this list of numbers: (0, 0, 1, 1, 2, 2, 3, 3, 4, 4). Show how we can use generating functions to determine how many ways there are to make a selection of 4 numbers having a sum of 10.

Each distinct number in the list becomes one factor in the generating function.
We need two variables: powers of x can count numbers in a selection, and powers of y can sum the numbers. Since there are 2 of each distinct value in the list, each factor of the generating function should allow us to take 0, 1 or 2 of these values.

This table can help determine the exponents for x and y.

Number in list	*If we take none*	*If we take one*	*If we take two*	*Factor*
0	*Count = 0* *Sum = 0*	*Count = 1* *Sum = 0*	*Count = 2* *Sum = 0*	$x^0y^0 + x^1y^0 + x^2y^0$
1	*Count = 0* *Sum = 0*	*Count = 1* *Sum = 1*	*Count = 2* *Sum = 2*	$x^0y^0 + x^1y^1 + x^2y^2$
2	*Count = 0* *Sum = 0*	*Count = 1* *Sum = 2*	*Count = 2* *Sum = 4*	$x^0y^0 + x^1y^2 + x^2y^4$
3	*Count = 0* *Sum = 0*	*Count = 1* *Sum = 3*	*Count = 2* *Sum = 6*	$x^0y^0 + x^1y^3 + x^2y^6$
4	*Count = 0* *Sum = 0*	*Count = 1* *Sum = 4*	*Count = 2* *Sum = 8*	$x^0y^0 + x^1y^4 + x^2y^8$

When we simplify, the generating function is:
$(1 + x + x^2)(1 + xy + x^2y^2)(1 + xy^2 + x^2y^4)(1 + xy^3 + x^2y^6)(1 + xy^4 + x^2y^8)$

The answer can be found by finding the coefficient of the term x^4y^{10}.
(Incidentally, Wolfram Alpha reports this answer is 5.)

109. Let L be this list of numbers: (1, 1, 2, 2, 3, 3). Explain how generating functions can be used to determine how many ways there are to:

a. Make a selection of 4 numbers.

   The generating function is $(1 + x + x^2)^3$. *We want the coefficient of* x^4.

b. Make a selection of numbers that adds up to 7.

   The generating function is $(1 + x + x^2)(1 + x^2 + x^4)(1 + x^3 + x^6)$. *We want the coefficient of* x^7.

c. Make a selection of 2 odd and 2 even numbers.

Let the exponent on x count the odds, and the exponent on y count the evens.
The generating function is $(1 + x + x^2)^2 (1 + y + y^2)$. We want the coefficient of x^2y^2.

An alternative approach: Since x and y never appeared in the same factor, we could separate the variables. It is as if we are making two selections: Select two odds from (1, 1, 3, 3), and select two evens from (2, 2). So, we would have two separate generating functions problems, and we multiply our answers.

First, find the coefficient of x^2 in $(1 + x + x^2)^2$.
Next, find the coefficient of y^2 in $(1 + y + y^2)$. [That should be no mystery.]
The product of these coefficients is our answer.

110. Explain how generating functions can be used to determine how many 5-digit numbers have a sum of digits of 20. (Note: the first digit of the number cannot be 0.)

Each digit is represented by a factor in our generating function. Selecting a digit means picking a number from 0 to 9 (or 1 to 9 in the case of the first digit).
Our generating function is:
$(x + x^2 + x^3 + \ldots + x^9)(1 + x + x^2 + \ldots + x^9)^4$.
We want the coefficient of x^{20}.

111. Show how generating functions can be used to solve this problem. We have 5 red apples, 5 yellow apples, and 4 green apples. How many ways are there to select 7 apples, such that we pick an odd number of each color?

Odd number of up to 5 red: $(x + x^3 + x^5)$
Odd number of up to 5 yellow: $(x + x^3 + x^5)$
Odd number of up to 4 green: $(x + x^3)$
Multiply these factors.
We want the coefficient of x^7 in $(x + x^3 + x^5)^2 (x + x^3)$.

112. A bowl contains 5 apples, 5 bananas, 4 oranges and 4 pears. Assume that fruits of the same type are identical. Show how we can use generating functions to answer this question: how many ways can we make a selection of 10 fruits so that we have at least 2 apples, no more than 3 bananas, an even number of oranges, and an odd number of pears?

Each type of fruit is represented by one factor in the generating function.
We use the constraints to tell us which terms to include for each factor.
The total number of each fruit tells us the highest possible exponent for each factor.

Apples: We want 2, 3, 4 or 5.
Bananas: We want 0, 1, 2 or 3.
Oranges: We want 0, 2 or 4.
Pears: We want 1 or 3.

The generating function is $(x^2 + x^3 + x^4 + x^5)(1 + x + x^2 + x^3)(1 + x^2 + x^4)(x + x^3)$.
We want the coefficient of x^{10}.
(Incidentally, Wolfram Alpha reports this answer is 14.)

113. Suppose you are planning a wedding reception and need to distribute the 12 bottles in a case of champagne among 4 dining room tables. How many ways can it be done under each of the following scenarios?

a. There are no restrictions.

Ball-in-urn with 12 identical objects and 4 categories. Answer: C(15, 3).

b. Each table must receive at least one bottle.

Four bottles have already been allocated to the tables, so we only have the freedom to distribute the remaining 8 bottles. Use the ball-in-urn formula with n = 8 and r = 4, and we obtain C(11, 3).

c. Each table may receive no more than 3 bottles.

We can start with our unrestricted answer we found in (a), and then subtract the illegal cases. The bad cases are where a table receives 4 or more bottles. But we should also be aware that we have a problem of inclusion/exclusion, because multiple tables could receive 4+ bottles. So, here is the plan:

Answer = (No restriction) – 4(one table gets 4+ bottles) + 6(two tables get 4+ bottles) – 4(three tables get 4+ bottles).
No restriction = BallUrn(12, 4) = C(15, 3).
One specific table gets 4+ bottles = BallUrn(8, 4) = C(11, 3)
Two specific tables get 4+ bottles = BallUrn(4, 4) = C(7, 3)
Three specific tables get 4+ bottles = BallUrn (0, 4) = C(3, 3).

Total answer: C(15, 3) – 4 C(11, 3) + 6 C(7, 3) – 4 C(3, 3).
*Incidentally, this works out to: 455 – 4*165 + 6*35 – 4*1 = 1*
Isn't it interesting that the answer simplifies to 1? If you think about it, this is not surprising. If no table can receive more than 3 bottles, then all tables must receive exactly 3 bottles. There is only one way to ensure that all tables receive an exact number of bottles.

If we approach this problem using generating functions, the answer might come out more readily. Each table gets its own factor in the generating function. We can have up to 3 bottles at each table, so a factor would look like this: $(1 + x + x^2 + x^3)$. And we have 4 tables, so we multiply four factors (which are all the same) to obtain our generating function: $(1 + x + x^2 + x^3)^4$. We want the coefficient of x^{12}. Right away we notice that x^{12} is the highest order term of the expansion, and its coefficient has to be 1.

d. Each table may receive no more than 4 bottles.

We can do this problem the same way as the previous part. It turns out like this:
No restriction: BallUrn(12, 4) = C(15, 3)
One specific table gets 5+ bottles = BallUrn(7, 4) = C(10, 3)
Two specific tables get 5+ bottles = BallUrn(2, 4) = C(5, 3)
It's not possible for 3 tables to get 5+ bottles if there are only 12 total.
So, this answer works out to C(15, 3) – 4 C(10, 3) + 6 C(5, 3).
*Numerically, it is 455 – 4*120 + 6*10 = 35*

Just for fun, let's also solve this with generating functions. Maybe you would prefer to do it this way from the start. Each table gets its own factor. And because the number of bottles at each table ranges from 0 to 4, inclusive, each factor looks like this: $(1 + x + x^2 + x^3 + x^4)$. So, we want the coefficient of x^{12} in $(1 + x + x^2 + x^3 + x^4)^4$.

According to Wolfram Alpha, this generating function simplifies to:
$1 + 4x + 10x^2 + 20x^3 + 35x^4 + 52x^5 + 68x^6 + 80x^7 + 85x^8 + 80x^9 + 68x^{10} + 52x^{11} + 35x^{12} + 20x^{13} + 10x^{14} + 4x^{15} + x^{16}$.
Good news! The coefficient of x^{12} is 35, and this matches our answer we got when we solved the problem using the ball-in-urn approach.

e. The first table must have no more than 1 bottle, the second table must receive at least 1 bottle, the third table must receive 0, 2 or 4 bottles, and the last table must receive either 2 or 3 bottles?

Let's solve using a generating function. The generating function has a factor for each table.
First table: $(1 + x)$
Second table: $(x + x^2 + x^3 + \ldots)$
Third table: $(1 + x^2 + x^4)$
Fourth table: $(x^2 + x^3)$

The second factor does not need to be an infinite series. We can stop at x^{12} since we know that there is a total of 12 bottles. In fact, if we look at the minimum requirement for the fourth table, we realize that the second table cannot get more than 10 bottles. So, our generating function is:
$(1 + x)(x + x^2 + x^3 + \ldots + x^{10})(1 + x^2 + x^4)(x^2 + x^3)$
And we want the coefficient of x^{12}.

You could stop there. But let me show you the simplified form of the generating function from Wolfram Alpha so we can look at what the numerical answer happens to be:
$x^3 + 3x^4 + 5x^5 + 7x^6 + 9x^7 + 11x^8 + 12x^9 + 12x^{10} + 12x^{11} + 12x^{12} + 11x^{13} + 9x^{14} + 7x^{15} + 5x^{16} + 3x^{17} + x^{18}$. *(Interestingly, despite our unusual requirements, the coefficients happen to be symmetric, probably because all of our factors were arithmetic sequences) Thus, our answer is 12.*

114. Use generating functions to solve this problem: In how many ways can a committee of 7 people be chosen out of a larger group of 16?

Each of the 16 people is distinct. Each person can be chosen or not chosen. So our generating function is $(1 + x)^{16}$, and we want the coefficient of x^7.

115. A bag contains 60 nickels, 20 dimes and 20 quarters. Assuming that the coins of any one denomination are indistinguishable, in how many ways can 10 coins be selected from the bag?

a. Solve this problem using the ball-in-urn approach.

Note that the total numbers of nickels, dimes and quarters given in the problem are only here to assure us that we won't run out of a denomination when we select 10 coins. We have 10 identical objects and 3 categories. So use BallUrn(10, 3) = C(12, 2).

b. Solve this problem using generating functions.

Each denomination is a factor in the generating function.
For the nickels, we can write $(1 + x + x^2 + \ldots + x^{60})$. *But in reality we won't be interested in more than 10 nickels so we can stop at* x^{10}. *We can handle the dimes and quarters the same way. Thus, our generating function is* $(1 + x + x^2 + \ldots + x^{10})^3$.
We want the coefficient of x^{10} *in this generating function.*

According to Mathematica (or Wolfram Alpha), our generating function simplifies as follows: $1 + 3x + 6x^2 + 10x^3 + 15x^4 + 21x^5 + 28x^6 + 36x^7 + 45x^8 + 55x^9 + 66x^{10} + 75x^{11} + 82x^{12} + 87x^{13} + 90x^{14} + 91x^{15} + 90x^{16} + \ldots + x^{30}$. *So, we see that the coefficient of* x^{10} *is 66. This matches our answer in part (a) because C(12, 2) = 12*11/2 = 66 also.*

A note of caution: When creating this generating function, we had assumed a maximum of 10 of each denomination. So, we should not look at higher terms than x^{10} *if we want to use this generating function to solve similar problems. It can only be used for up to 10 coins being selected. For more than 10 coins, this generating function could only be used if we accept the restriction of 10 per denomination.*

c. Redo parts (a) and (b) assuming that we select 25 coins.

We have 25 identical objects and 3 categories. Use BallUrn(25, 3) = C(27, 2).
However, two of the categories have a limit of 20. So, we need to subtract those cases. The case of selecting 21 or more dimes would allow us to freely choose 4 coins. This has BallUrn(4, 3) = C(6, 2) possibilities. And the same is true for the quarters.
Answre: C(27, 2) – 2 C(6, 2).

The generating function again has three factors, but each factor needs to go up to at least the x^{25} *term. But wait, the dimes and quarters are limited to 20, so these factors can only go up to* x^{20}. *Okay, let's not try to simplify the generating function anymore. Let's have the highest power of each factor be the actual number of coins of each denomination. (Although you could let the number of nickels stop at 25, since we only want 25 total coins.) So, the generating function is* $(1 + x + x^2 + \ldots + x^{60})(1 + x + x^2 + \ldots + x^{20})^2$. *Here, we want the coefficient of* x^{25}.

d. Redo parts (a) and (b) assuming that we select 50 coins.

This is similar to part (c), but we could run out of both dimes and quarters. These cases would be counted twice when we count the number of ways of running out of dimes and quarters separately. In other words, it's a question of inclusion/exclusion.

To simulate running out of dimes [or quarters], we want 21 or more of them, leaving 29 coins to be freely chosen. To simulate running out of both dimes and quarters, we want 21 of each, leaving only 8 coins to be freely chosen.

General form of answer: (unrestricted case) – (run out of dimes) – (run out of quarters) + (run out of both).
Unrestricted case: BallUrn(50, 3) = C(52, 2)
Run out of dimes: BallUrn(29, 3) = C(31, 2)
Run out of quarters: same as run out of dimes = C(31, 2)
Run out of both: BallUrn(8, 3) = C(10, 2)

Final answer: C(52, 2) – 2 C(31, 2) + C(10, 2)

Using generating functions, we can use the same one from the previous part. Look for the coefficient of x^{50} in $(1 + x + x^2 + \ldots + x^{60})(1 + x + x^2 + \ldots + x^{20})^2$.

116. How many ways are there to make 15 cents change from pennies minted in 1952 and 1959 and nickels minted in 1964?

a. Solve this problem using the ball-in-urn approach.

The number of coins of each type unlimited. But there is more than one way to come up with 15 cents. It depends on how many nickels we choose.

If no nickels: We have 15 identical objects and 2 categories. BallUrn(15, 2) = C(16, 1)
If 1 nickel: We have 10 identical objects and 2 categories. BallUrn(10, 2) = C(11, 1)
If 2 nickels: We have 5 identical objects and 2 categories. BallUrn(5, 2) = C(6, 1)
If 3 nickels: We have 0 identical objects and 2 categories. BallUrn(0, 2) = C(1, 1)

Answer: C(16, 1) + C(11, 1) + C(6, 1) + C(1, 1)
This can easily simplify to: 16 + 11 + 6 + 1 = 34.

b. Solve this problem using generating functions.

Let's use a factor for each type of coin. The exponent on x will be the number of cents we want to keep track of. We want the coefficient of x^{15} in this generating function: $(1 + x^5 + x^{10} + x^{15})(1 + x + x^2 + \ldots + x^{15})^2$.

Note: You could let each factor go on forever, but it's sufficient in this problem to stop each factor at x^{15} since we know we only want 15 cents. The generating function here could then be re-used for smaller amounts of money. But if you want to handle larger amounts, you will need more terms in each factor.

According to Mathematica, this simplifies to: $1 + 2x + 3x^2 + 4x^3 + 5x^4 + 7x^5 + 9x^6 + 11x^7 + 13x^8 + 15x^9 + 18x^{10} + 21x^{11} + 24x^{12} + 27x^{13} + 30x^{14} + 34x^{15} + \ldots$

So, we see that the coefficient of x^{15} is 34, which matches our answer in part (a).

117. Using the fact that $(x + x^2 + x^3 + x^4 + x^5 + x^6)^4$ equals $x^4 + 4x^5 + 10x^6 + 20x^7 + 38x^8 + 56x^9 + 80x^{10} + 104x^{11} + 125x^{12} + 140x^{13} + 146x^{14} + 140x^{15} + 125x^{16} + 104x^{17} + 80x^{18} + 56x^{19} + 35x^{20} + 20x^{21} + 10x^{22} + 4x^{23} + x^{24}$, what question can we ask about rolling dice to which the answer is 125?

If we roll four dice, how many ways are there of obtaining a sum of 12? Alternatively, we could ask for a sum of 16, since the 125 coefficient appears in two places.

118. How many terms are in the expansion of $(1 + ax + a^2x^2)(1 + bx)(cx + c^2x^2 + c^3x^3)(1 + d^3x^3 + d^5x^5)$? What combinatorical questions can this generating function answer?

*The number of terms is 3*2*3*3. The reason why is that each time we want to write down a new term, we select a different combination of terms from each factor. So, this is just an application of* *the multiplication rule. Also note that the individual terms won't combine, because of the different coefficient letters used in each factor.*

With this generating function, we could list all of the ways that we could do the following experiment: We want to select some number of a's, b's, c's and d's. We must select these letters subject to all of the following requirements:
Zero, one or two a's
Zero or one b
One, two or three c's
Zero, three or five d's

The generating function by itself is usually not enough; we need to focus on one of its terms. In this generating function, the coefficient of x^k will tell us all of the possible ways of collecting a total of k letters in our selection.

119. What question about rolling dice is answered by seeking the coefficient of x^{25} in this generating function?

$$\left(\sum_{i=1}^{10} x^i\right)^6$$

We have six factors, which suggests that we have six dice. Each factor contains terms from x^1 to x^{10}, which suggests that each die has faces numbered 1 to 10. So, the question is asking for the number of ways we can roll six 10-sided dice to obtain a sum of 25.

120. In the online game Apterous there is a numbers round where we have 24 numbered tiles face down, and six of these tiles are to be selected at random. The 24 tiles consist of the following: two each of the "small" values 1-10, plus one each of the "large" values 25, 50, 75 and 100. How many ways are there to select…

a. Any six numbers?

We need a factor for each distinct number we can select. The numbers 1-10 can each be selected up to 2 times. The numbers 25, 50, 75 and 100 can be selected up to once. Therefore, the generating function is $(1 + x + x^2)^{10} (1 + x)^4$, and we want the coefficient of the x^6 term because we want to select 6 numbers.

According to Wolfram Alpha, the desired coefficient is 13,243.

b. Six small numbers?

We can use a generating function to help us make selections of small numbers. The large numbers are unique, so we can take the easy route and use combinations for those.

The generating function for the small numbers is $(1 + x + x^2)^{10}$. Since we want 6 of these, we need the coefficient of x^6. We want none of the large numbers, and this selection can be made in C(4, 0) ways. And then we multiply these two answers.

I used Wolfram Alpha to work out the generating function coefficient computations throughout this problem. It turns out that the coefficient of x^6 in $(1 + x + x^2)^{10}$ is 2850, and C(4, 0) = 1. So our answer works out to 2850.

It's far more important to know how to set up the problem than getting the numerical result.

c. Five small and one large?

*For the small numbers, we need the coefficient of x^5 in $(1 + x + x^2)^{10}$, which happens to be 1452. For the large numbers, we want C(4, 1), which equals 4. Our answer is 1452 * 4 = 5808.*

Note that this answer is numerically the largest of parts (b) through (f). This means that if someone were to pick 6 numbers completely at random, this is the most likely outcome. Given the overall distribution of tiles, twenty with small numbers and four with large (i.e. one-sixth of the tiles showing a large number), this result is not surprising. In the long run we'd expect exactly one large number on average.

d. Four small and two large?

For the small numbers, we need the coefficient of x^4 in $(1 + x + x^2)^{10}$, which is 615. For the large numbers, we want C(4, 2) = 6. Multiplying, we get an answer of 3690.

e. Three small and three large?

For the small numbers, we want the coefficient of x^3 in $(1 + x + x^2)^{10}$, which is 210. For the large numbers, we want C(4, 3) = 4. Multiplying, we get 840.

f. Two small and all four of the large?

For the small numbers, we want the coefficient of x^2 in $(1 + x + x^2)^{10}$, which is 55. For the large numbers, we want C(4, 4) = 1. Answer = 55.

g. Look at your answers so far, and verify that a = b+c+d+e+f. Why should we expect this to be true?

a = 13243
b+c+d+e+f = 2850 + 5808 + 3690 + 840 + 55 = 13243
They match! Case a asks for how many ways to select 6 numbers. Cases b-f enumerate based on how many of the 6 are large. The only cases that seem to be "missing" are having 5 or 6 large numbers, but there are only 4 large numbers from the original list. So, we have accounted for every possibility.

h. Six numbers whose sum is 40?

We need a different generating function for this problem, one that makes use of all of the possible number tiles. Let's use the power of x to keep track of the number of tiles, and the *power of y be the sum. Our generating function would be* $(1 + xy + x^2y^2)(1 + xy^2 + x^2y^4)(1 + xy^3 + x^2y^6)(1 + xy^4 + x^2y^8)(1 + xy^5 + x^2y^{10})(1 + xy^6 + x^2y^{12})(1 + xy^7 + x^2y^{14})(1 + xy^8 + x^2y^{16})(1 + xy^9 + x^2y^{18})(1 + xy^{10} + x^2y^{20})(1 + xy^{25})(1 + xy^{50})(1 + xy^{75})(1 + xy^{100})$*. We want the coefficient of* x^6y^{40}*.*

i. Four odd and two even numbers?

Let the exponent of x be the number of odd, and the exponent of y be the number of even numbers we select. As before, each distinct number (1, 2, 3, …) is a factor in our generating function.

The odd numbers are 1, 3, 5, 7, 9, 25, 75. The small numbers may be selected up to twice. For example, the factor we would use for each of the numbers 1, 3, 5, 7 and 9 would be $(1 + x + x^2)$*. For 25 and 75, the factor would just be* $(1 + x)$*. Multiplying all these factors, our generating function needs this:* $(1 + x + x^2)^5 (1 + x)^2$*.*

The situation for the even numbers 2, 4, 6, 8, 10, 50 and 100 is analogous. The only difference is that we are counting evens instead of odds. These factors give us $(1 + y + y^2)^5 (1 + y)^2$*.*

Therefore, the generating function to solve this problem is $(1 + x + x^2)^5 (1 + x)^2 (1 + y + y^2)^5 (1 + y)^2$*. We want the coefficient of* x^4y^2*. According to Wolfram Alpha, this coefficient is 3120.*

Actually, there is a slightly easier approach to this problem. Notice that the selection of odds and evens can be done independently. Notice in the above generating function we never wrote x and y in the same factor. If we ***separate the variables****, we wind up with two simple generating functions instead of one ugly one. We can solve the two little problems, and just multiply our answers by virtue of the multiplication rule.*

Choosing just the odds: coefficient of x^4 *in* $(1 + x + x^2)^5 (1 + x)^2$*, which is 120.*
Choosing just the evens: coefficient of y^2 *in* $(1 + y + y^2)^5 (1 + y)^2$*, which is 26.*
(And because the two selections are different, you could have used the letter x instead of y.)
As expected, 120 times 26 = 3120.

121. Determine whether the following functions are one-to-one and/or onto.

a. The sum function defined as sum : $Z^2 \rightarrow Z$, sum (a, b) = a+b.

This function is not one-to-one, because we can find two ordered pairs that have the same functional value. In other words, we can find two different pairs of numbers that have the same sum. For example, (1, 5) and (2, 4) both sum to 6.

The function is onto. Every integer y can be expressed as the sum of two integers (a, b). Simply let a = y and b = 0. Then a+b = y+0 = y.

b. The function f defined as f: X → X where X = { 0, 1, 2, 3, 4 } and the rule is
f(x) = 4x mod 5

Let's write out all the ordered pairs of this function, since it's so small.
*For x = 0, 4*0 mod 5 = 0 mod 5 = 0*
*For x = 1, 4*1 mod 5 = 4 mod 5 = 4*
*For x = 2, 4*2 mod 5 = 8 mod 5 = 3*

For $x = 3$, $4*3 \bmod 5 = 12 \bmod 5 = 2$
For $x = 4$, $4*4 \bmod 5 = 16 \bmod 5 = 1$.
The function is this set of ordered pairs: {(0, 0), (1, 4), (2, 3), (3, 2), (4, 1)}.

This function is one-to-one because we see that every x maps to a different y.
The function is onto because every number 0-4 is in the range.

c. The function F for which the domain is (0+1)* and the codomain is { true, false }. The rule for F is: F(x) = true if x has an equal number of 0's and 1's; and F(x) = false otherwise.

F is not one-to-one. For example, we can find two different strings x and y where F(x) and F(y) are both true. In other words, we can find two different strings that both have an equal number of 0's and 1's. Let x = 01 and y = 10.

F is onto. To show this, we need to produce an x for which F(x) is true, and a y for which F(y) is false. Let x = 10 and let y = 0. The string 10 has an equal number of 0's and 1's. The string 0 has an unequal number of 0's and 1's.

d. The function s defined as s: $Z^+ \rightarrow Z^+$ where s(x) is the sum of digits of x.

The function s is not one-to-one. We can find two numbers x that have the same sum of digits. For example, consider 12 and 21. We see that s(12) = 3 and s(21) = 3.

The function is onto. Given any value for the sum of digits y, we can produce an input value x that has this sum of digits. The number x would be a concatenation of y ones. For example, if y = 7, then let x = 1,111,111.

e. The function f with domain $Z \times Z$ and co-domain Z, where $f(x, y) = |x| - |y|$. The vertical bars mean absolute value.

The function f is not one-to-one. We can find two ordered pairs (x, y) that have the same difference of their absolute values. For example, consider (5, 2) and (9, 6). Both have a difference of 3: $f(5, 2) = |5| - |2| = 3$, and $f(9, 6) = |9| - |6| = 3$.

The function is onto. Given any integer value z, we can produce an ordered pair (x, y) for which $|x| - |y| = z$.
If z is zero, we can choose x = 0 and y = 0. Then, $f(0, 0) = 0 = z$.
If z is positive or zero, we can chose x = z and y = 0. Then, $f(z, 0) = z$

If z is negative, we can choose x = 0 and y = z. Here, we need to be careful:
$f(0, z) = 0 - |z|$. Notice that $z < 0$ means that $|z| = -z$.
Therefore, $f(0, z) = 0 - (-z) = z$.

f. The function F whose domain is the set of binary strings. The co-domain is the set of non-negative integers. The rule of F is: F(x) = the number of zeros that appear in x.

F is not one-to-one. We can produce two different binary strings that have the same number of zeros. For example, 100 and 010. F(100) = 2 and F(010) = 2.

F is onto. Given any number of zeros y, we can produce a binary string that has this many zeros. Let x be the string having y zeros, and no ones. Clearly, x has y zeros.

122. How many onto functions are there mapping X to Y (i.e. X → Y) if X has 4 elements and Y has 3 elements?

First, let's find the total number of functions. Each of the 4 values in X has to choose among the 3 values of Y to be its functional value, or what the book calls its "image." Therefore, the number of functions is $3*3*3*3 = 3^4$.

We need to subtract away those functions that only assign to 2 of the 3 values in Y. There are C(3, 1) ways to choose the value in Y not in the range of the function. Then, we ask each of the values in X which of the 2 values is its functional value. Therefore, the number of illegal functions is $C(3, 1) * 2^4$.

While we were subtracting away some bad functions, we actually had some double counting. Some cases occurred twice. We have to add back these cases. It could be that a function leaves out two values in its range. There are C(3, 2) ways this could happen. And then, each value in X has just 1 value in Y to be its functional value. Thus, the number of functions here is $C(3, 2) * 1^4$.

The final answer is $3^4 - C(3, 1)\ 2^4 + C(3, 2)\ 1^4$.

123. Suppose set A has seven elements and set B has three elements. Consider all functions that use A as the domain and B as the co-domain.
   a. How many such functions exist?

      For each value in the domain, we need to choose one corresponding value in the co-domain. With no restrictions, this will be 3^7.

   b. How many of these functions are onto?

      To be onto, we have to exclude functions that only map to at most 2 of the values in the co-domain.
      How many functions map to at most two values in the co-domain? First, choose which two values: C(3, 2). Next, create the function – For each value in the domain, decide which of the two values in the co-domain will be the functional value. This can be done in 2^7 *ways. Note that some of these* 2^7 *functions will even map all inputs to a only single value in the co-domain.*

      Our final answer is $3^7 - C(3, 2)\ 2^7$.

124. Is the concatenation of binary strings a one-to-one operation? Is it onto?

It is not one-to-one. For example, the string 101 can be constructed by concatenating 10 with 1, or concatenating 1 with 01.

It is onto. Every string z can be expressed as the concatenation of two strings x and y. For example, let x = z and let y = ε.

125. Design finite automata that accept the following sets of bit strings (draw one FA for each part):

a. The input begins with 00.

State	*Input 0*	*Input 1*
→ Need 00	*Need 0*	*Reject*
Need 0	*Happy*	*Reject*
☺ Happy	*Happy*	*Happy*
Reject	*Reject*	*Reject*

b. The input ends with 00.

State	*Input 0*	*Input 1*
→ Need 00	*Need 0*	*Need 00*
Need 0	*Happy*	*Need 00*
☺ Happy	*Happy*	*Need 00*

c. The input has two consecutive zeros.

State	*Input 0*	*Input 1*
→ Need 00	*Need 0*	*Need 00*
Need 0	*Happy*	*Need 00*
☺ Happy	*Happy*	*Happy*

d. The input does not have two consecutive zeros.

This is the same as the previous part, except that the accept states are "Need 00" and "Need 0" and the reject state is "Happy."

e. The input begins with 0 and contains exactly two 1's.

State	*Input 0*	*Input 1*
→ Need 0, 11	*Need 11*	*Reject*
Need 11	*Need 11*	*Need 1*
Need 1	*Need 1*	*Happy*
☺ Happy	*Happy*	*Reject*
Reject	*Reject*	*Reject*

126. Let L be the set of all binary strings that begin with 1 and end with 0.

a. How many words in L have a length of 16?

We may choose anything for the middle 14 bits. Answer: 2^{14}.

b. Write a regular expression for L.

*1(0 + 1)*0.*

c. Write a grammar for L.

S → 1A0
A → ε | A0 | A1

d. Draw a finite automaton that recognizes L.

State	Input 0	Input 1
→ Need 1,0	Reject	Need 0
Need 0	Happy	Need 0
☺ Happy	Happy	Need 0
Reject	Reject	Reject

127. Let L be the set of binary strings that begin with 0 and contains exactly one 1.

a. Draw a finite automaton for L.

State	Input 0	Input 1
→ Need 0,1	Need 1	Reject
Need 1	Need 1	Happy
☺ Happy	Happy	Reject
Reject	Reject	Reject

b. Write a regular expression for L.

*00*10**

128. Let S be the set of binary strings that begin with 0 and end with 11.

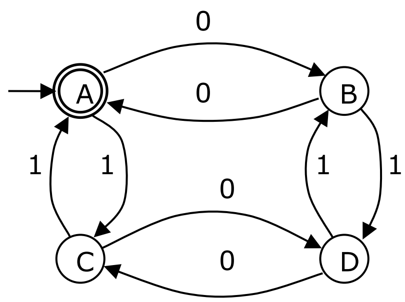
a. Write a regular expression for S

0 (0 + 1) 11*

Write a grammar for S.

S → 0A11
A → ε | A0 | A1

129. What is the set of binary strings that is accepted by the following finite automaton?



All binary strings that have an even number of zeros and an even number of ones.

130. Explain why it is not possible to design a finite automaton that accepts the set of bit strings that have the same number of 0's and 1's.

The input could begin with a lot of 0's. Or it could start with a lot of 1's. We would need states to allow the machine to keep track of how many unmatched 0's or 1's have so far been read in. Intuitively, it seems that we would then need an unbounded number of states. When you study computational theory, you will learn more precise ways to show that this sort of FA cannot exist.

131. Consider this definition of a Boolean expression:

A Boolean expression is defined by the following rules.

I. A statement variable (e.g. p, q, r, …) standing alone is a Boolean expression.
II. If E is Boolean expression, then ~(E) is a Boolean expression.
III. If E and F are Boolean expression, then so are (E) + (F) and (E) • (F).
IV. A string of symbols is a Boolean expression if and only if it derives from finitely many applications of rules I, II, III. (i.e. Nothing else is a Boolean expression.)

Determine if the following strings of symbols satisfy the above definition.

a. ((p) + (q)) + ((p) • (r))

By Rule I, we see that p, q and r are Boolean expressions.
By rule III, (p) + (q) and (p) • (r) are Boolean expressions.
Finally, by rule III again, ((p) + (q)) + ((p) • (r)) is a Boolean expression.

b. ((p) + q)

This is not a Boolean expression. There is no rule that allows a + to be followed immediately by a statement letter.

c. ((p) + (q) + ~(r))

This is not a Boolean expression. There is no rule that allows a + to be followed immediately by a ~.

132. Let's write explicit formulas for the sequence: 1, 5, 1, 5, 1, 5, ….

a. Write a formula that uses (–1) raised to an integer power.

Note that –1 raised to an even power equals +1, and –1 raised to an odd power equals –1.

The numbers in the sequence are all 2 away from 3. So, we can write the number 1 of the form 3 + 2(–1), and the number 5 of the form 3 + 2(+1).

So, it appears that our formula for a_n should be $3 + 2(-1)^n$ or $3 + 2(-1)^{n+1}$. Take note of the first term in the sequence. If n = 1, then $3 + 2(-1)^n = 1$, which is correct.

Therefore, our answer is $a_n = 3 + 2(-1)^n$.

b. Write a formula that uses the cosine of an integer multiple of pi.

Note that $\cos(0\pi) = 1$, $\cos(1\pi) = -1$, $\cos(2\pi) = 1$, $\cos(3\pi) = -1$, etc.

Our formula should either be $3 + 2\cos(n\pi)$ or $3 + 2\cos((n+1)\pi)$. Which is right? Look at the first term. When n = 1, the formula should evaluate to 1. Using our first formula, $3 + 2\cos(1\pi) = 3 - 2 = 1$, which is correct.

Thus, our answer is $a_n = 3 + 2\cos(n\pi)$.

133. For this sequence of numbers: 10, 5, 0, –5, –10, –15, …, assume the first number of the sequence is called a_1.

a. Write a recursive definition of the sequence.

$a_1 = 10$, and for all $n \geq 2$, $a_n = a_{n-1} - 5$.

b. Write an explicit definition of the sequence.

Think of the question as asking for the equation of a line. The slope is –5, and the y-intercept, the hypothetical a_0 term, is 15.

$a_n = 15 - 5n$.

134. Write recursive definitions for:

a. The sequence of positive odd integers.

$a_1 = 1$
$a_n = a_{n-1} + 2$, for $n \geq 2$.

b. This sequence of perfect squares: 0, 1, 4, 9, 16, …

Note that the perfect squares differ by odd numbers, so we can use our answer from part (a)!
$b_0 = 0$

$b_n = b_{n-1} + a_n$, *for* $n \geq 1$.

c. The sequence 17, −22, 27, −32, 37, −42, 47, −52, 57, …

$c_1 = 17$
$c_n = (|c_{n-1}| + 5)(-1)^{n+1}$, $n \geq 2$.
Be careful where you put the absolute value bars.

d. The sequence −3, 13, −23, 33, −43, 53, −63, 73, …

$d_1 = -3$
$d_n = (|d_{n-1}| + 10)(-1)^n$, $n \geq 2$.

135. Suppose a sequence of numbers is defined by the explicit formula $a_n = n^2 - n$ for $n \geq 1$. Write a recursive formula for this sequence.

Let's figure the first few terms to notice the recursive pattern.

n	1	2	3	4	5	6
a_n	0	2	6	12	20	30

To obtain each term, what must we do with the previous term?
In other words, what did we do to the 2 to obtain 6?
What did we do to the 6 to obtain 12?
What did we do to the 12 to obtain 20?
What did we do to the 20 to obtain 30?
We see that the new term is the previous term plus twice the previous term number. The recursive formula is:

$a_1 = 0$
$a_n = a_{n-1} + 2(n - 1)$

136. For the following sets of binary strings, write both a regular expression and a grammar (i.e. recursive definition) for the set.

a. L = the set of binary strings that begin with 0.

Regular expression: $0(0 + 1)^*$

Grammar: $S \rightarrow 0A$
$A \rightarrow \varepsilon \mid A0 \mid A1$

b. L = the set of binary strings that have two consecutive 1's.

Regular expression: $(0 + 1)^*11(0 + 1)^*$

Grammar: $S \rightarrow A11A$

A → ε | A0 | A1

c. L = the set of binary strings that begin with 0, and contain two consecutive 1's.

*Regular expression: 0(0 + 1)*11(0 + 1)**

Grammar: *S → 0A11A*
A → ε | A0 | A1

d. L = the set of binary strings that begin and end with 0, and contain two consecutive 1's.

*Regular expression: 0(0 + 1)*11(0 + 1)*0*

Grammar: *S → 0A11A0*
A → ε | A0 | A1

e. L = the set of binary strings that begin with 0 and end with 1.

*Regular expression: 0(0 + 1)*1*

Grammar: *S → 0A1*
A → ε | A0 | A1

f. Write a grammar only for this language: L = the set of binary strings consisting of n 0's followed by 2n 1's.

S → ε | 0S11

137. Consider the following definition of integer exponentiation:

exp (a, n) equals 1 if n = 0; or a * exp (a, n – 1) if n >= 1.

Give another recursive definition that would require fewer recursive calls for large values of n.

Let's split up the recursive case into even/odd cases.

exp(a, n) equals:

1	*if n = 0*
$(exp(a, n/2))^2$	*if n is even*
a exp(a, n – 1)	*if n is odd*

The squaring in the even case can be accomplished by multiplying the return value by itself, so it's not really a circular definition.

138. Ackermann's function is a recursive function defined by the following rules.
A(0, y) = 1
A(1, 0) = 2

A(x, 0) = x + 2 for $x \geq 2$
A(x, y) = A(A(x − 1, y), y − 1)

Compute the value of A(2, 2). In your solution, you distinguish when you are using a base rule or a recursive rule.

The first 3 rules are bases cases. Only rule 4 is recursive.
A(2, 2) = A(A(1, 2), 1) by rule 4.
A(1, 2) = A(A(0, 2), 1) by rule 4.
A(0, 2) = 1 by rule 1.
Now we know that A(1, 2) = A(1, 1).
A(1, 1) = A(A(0, 1), 0) by rule 4.
A(0, 1) = 1 by rule 1.
Now we know that A(1, 1) = A(1, 0).
A(1, 0) = 2 by rule 2.
Now we know that A(1, 1) = 2.
Now we know that A(1, 2) = 2.
Now we know that A(2, 2) = A(2, 1).
A(2, 1) = A(A(1, 1), 0) by rule 4.
We can repeat earlier steps to see that A(1, 1) = 2.
Now we know that A(2, 1) = A(2, 0).
A(2, 0) = 4 by rule 3.
Now we know that A(2, 1) = 4
Now we know that A(2, 2) = 4, and this is our final answer.

139. Let f(x) be defined for all real numbers as follows.
f(x) = 0, if $x \leq 0$
f(x) = 1 + f(log(x)), if $x > 0$
Assume that the log is taken to the base 10. For which values of x does f(x) return 3?

This function is asking how many times we need to take a common logarithm until the answer is zero or negative. It's similar to the log function, which asks how many times a log to the base 2 can be taken until the result is 1 or less. This type of function grows very slowly.*

The base case is triggered when the argument x is 0 or negative. So, 0 is the largest number for which f(x) equals 0.
Notice that f(1) = 1. In evaluating f(1), we recursively call with f(0), and the argument 0 is the largest value of x that triggers the base case. So, 1 is the largest x making f(x) = 1.
Next, consider f(10). After two recursive calls, we reach f(0). So, 10 is the largest x that makes f(x) = 2.
By continuing this reasoning, we find that 10^{10} is the largest x making f(x) = 3. Therefore the set of numbers resulting in f(x) = 3 is $10 < x \leq 10^{10}$.

140. Suppose Fibonacci numbers are calculated by the following function. How many times is F called when we evaluate F(5)?

```
int F(int n)
{
  if (n <= 2)
    return 1;
```

```
  else
    return F(n-1) + F(n-2);
}
```

main calls F(5) // 1 call to F
F(5) calls F(4) and F(3) // 2 more calls to F
F(4) calls F(3) and F(2) // 2 more calls to F
F(3) calls F(2) and F(1) // 2 more calls to F
F(2) immediately returns
F(1) immediately returns
F(2) immediately returns
F(3) calls F(2) and F(1) // 2 more calls to F
F(2) immediately returns
F(1) immediately returns

There are 9 total calls to F.

141. Let f(n) represent the Fibonacci sequence. Prove the following for all positive integers n.

a. $f(n + 4) = 3 f(n + 2) - f(n)$ [Hint: don't use induction for this part]

Let's apply the recursive definition of Fibonacci to rewrite f(n+4):

$$f(n+4) = f(n+3) + f(n+2)$$
$$= (f(n+2) + f(n+1)) + f(n+2)$$
$$= 2 f(n+2) + f(n+1)$$

We can also rewrite f(n+1). Notice that:
$f(n+2) = f(n+1) + f(n)$, so that
$f(n+1) = f(n+2) - f(n)$. Now substitute in our formula for f(n+4) to obtain:

$$f(n+4) = 2 f(n+2) + (f(n+2) - f(n)$$
$$= 3 f(n+2) - f(n)$$

Note that this formula is useful because it allows us to advance through the Fibonacci sequence twice as fast because the subscripts that it uses are n, n+2 and n+4, which are all two apart instead of being one apart in the traditional formula. Hmmm, I wonder if we could compose a similar formula for f(n+6) in terms of just f(n+3) and f(n). ☺

b. $f(n) < (7/4)^n$

The base case is where n = 1. Then, f(1) = 1 and $(7/4)^1 = 7/4$. The assertion is true because $1 < 7/4$.

Next, let's assume that the inequality holds for all positive integers up to and including k. (Here we are using the strong form of induction, rather than just assuming the original statement is true just for k.)
This means that we are assuming:

$f(k - 1) < (7/4)^{k-1}$ and $f(k) < (7/4)^k$

We need to show that $f(k + 1) < (7/4)^{k+1}$.

Consider the number f(k + 1). By the definition of the Fibonacci function,

$f(k + 1) = f(k) + f(k - 1)$

Using what we know about $f(k)$ and $f(k - 1)$, let's substitute:

$f(k + 1) < (7/4)^k + (7/4)^{k-1}$

And we can simplify the right side of this inequality as follows:

$f(k + 1) < (7/4)^{k-1} (7/4 + 1)$
$f(k + 1) < (7/4)^{k-1} (11/4)$

What is the "difference" between $(7/4)^{k+1}$ and $(7/4)^{k-1}$? Two factors of (7/4), which equals 49/16. Is 49/16 greater than 11/4? Well, 49/16 = 3.0625 and 11/4 = 2.75. So, we can say that $11/4 < 49/16$ and:

$f(k + 1) < (7/4)^{k-1} (11/4) < (7/4)^{k-1} (49/16)$
$f(k + 1) < (7/4)^{k-1} (49/16)$

and this last inequality is equivalent to:

$f(k + 1) < (7/4)^{k+1}$

which is the result we needed to verify the inductive step. We're done! Since the base case and inductive cases are verified, the original assertion must hold for all positive integers by the strong form of induction.

c. The sum of the first n Fibonacci numbers is equal to $f(n+2) - 1$.

Base case: When $n = 1$, this statement is saying that the sum of just $f(1)$ is $f(3) - 1$. Since $f(1) = 1$ and $f(3) = 2$, this equation becomes $1 = 2 - 1$, which is true.

Inductive case: Assume that the sum of the first k Fibonacci numbers is $f(k+2) - 1$. Let's derive a formula for the sum of the first k+1 Fibonacci numbers. It would be:

Sum of first k+1 = Sum of first k + $f(k+1)$
$= f(k+2) - 1 + f(k + 1)$
$= f(k+2) + f(k+1) - 1$
And note that the recursive definition of the Fibonacci sequence says that $f(k+3) = f(k+2) + f(k+1)$, so we can write the right side of the equation as
$= f(k+3) - 1$
So, we have arrived at the statement $P(k+1)$.

Since $P(1)$ is true and $P(k)$ implies $P(k+1)$, therefore the original statement $P(n)$ is true for all positive integers n.

142. In each of the following cases, use induction to show that the indicated explicit formula solves the given recurrence relation.

a. $a_n = 2^n + 2(3^n)$ solves the recurrence relation given by: $a_0 = 3$, $a_1 = 8$ and $a_n = 5\, a_{n-1} - 6\, a_{n-2}$ for $n \geq 2$.

Base cases:
If $n = 0$, the explicit formula gives $a_0 = 2^0 + 2(3^0) = 1 + 2 = 3$
If $n = 1$, the explicit formula gives $a_1 = 2^1 + 2(3^1) = 2 + 2*3 = 8$

Inductive step: Assume that the explicit formula is correct up through $n = k$. Then use the recursive formula to determine a formula for a_{k+1} and algebraically check that it works out to the same expression we would get if we substitute k+1 into the explicit formula:

We assume that $a_k = 2^k + 2(3^k)$ and $a_{k-1} = 2^{k-1} + 2(3^{k-1})$.
The recursive formula tells us how to write a_{k+1}:

$a_{k+1} = 5a_k - 6a_{k-1}$
$= 5\,(2^k + 2(3^k)) - 6\,(2^{k-1} + 2(3^{k-1}))$
$= 5\,(2^k) + 10\,(3^k) - 6\,(2^{k-1}) - 12\,(3^{k-1})$
$= 5\,(2^k) + 10\,(3^k) - 3\,(2^k) - 4\,(3^k)$
$= 2\,(2^k) + 6\,(3^k)$
$= 2^{\,k+1} + 2\,(3^{\,k+1})$
This last expression is what we would obtain by substituting k+1 into the explicit formula.

Since the formulas match for $n = 0$ and $n = 1$, and assuming they match through k allows us to conclude they also match at k+1, then the formulas match for all positive integers by the principle of mathematical induction.

b. $a_n = 4(2^n) - 3$ solves the recurrence relation given by: $a_0 = 1$, $a_n = 2a_{n-1} + 3$ for $n \geq 1$.

Base case:
If $n = 0$, the explicit formula gives $a_0 = 4(2^0) - 3 = 4 - 3 = 1$.

Inductive step: Assume that the formulas match at $n = k$. Then, we need to show they also match at $n = k+1$. Use the recursive formula to develop an expression for a_{k+1}. Then show algebraically it is equivalent to what we would obtain if we substitute k+1 into the explicit formula.

We assume that $a^k = 4(2^k) - 3$.
The recursive formula tells us how to write a_{k+1}:

$a_{k+1} = 2a_k + 3$
$= 2\,(4(2^k) - 3) + 3$
$= 8\,(2^k) - 6 + 3$
$= 8\,(2^k) - 3$
$= 4\,(2^{k+1}) - 3$
This last expression is what we would obtain if we substitute k+1 into the explicit formula.

Since the formulas match when $n = 0$, and assuming they match at $n = k$ allows us to conclude they also match at $n = k+1$, then we can say the formulas match for all positive integers by the principle of mathematical induction.

143. Consider the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2}$, with $a_0 = 10$ and $a_1 = 41$.

a. Write an explicit formula for a_n.

The parallel quadratic equation is $x^2 - 7x + 10 = 0$. This can be factored as $(x - 5)(x - 2) = 0$, so the roots are $x = 5$ and $x = 2$. This means that the general form of our solution is $a_n = c_1 5^n + c_2 2^n$. Next, we substitute $n = 0$ and $n = 1$ using the given base cases in order to solve for c_1 and c_2.

$c_1 5^n + c_2 2^n = a_n$
If $n = 0$, this equation becomes: $c_1 5^0 + c_2 2^0 = a_0$
$c_1 + c_2 = 10$
If $n = 1$, this equation becomes: $c_1 5^1 + c_2 2^1 = a_1$
$5c_1 + 2c_2 = 41$

We now have a system of 2 equations in 2 variables. If we multiply the first equation by –2 and add it from the second equation, we can eliminate c_2 as follows:

$-2c_1 - 2c_2 = -20$
$5c_1 + 2c_2 = 41$

$3c_1 = 21$
$c_1 = 7$

And if we substitute $c_1 = 7$ into our earlier equation $c_1 + c_2 = 10$, then we see that $c_2 = 3$.

Now we can write our final answer. $a_n = 7 (5^n)+ 3 (2^n)$

b. Use induction to verify that your explicit formula is correct.

First, we need to substitute 0 and 1 in for n into the explicit solution to see that they match our base cases.

If $n = 0$, $a_0 = 7 (5^0)+ 3 (2^0) = 7 + 3 = 10$
If $n = 1$, $a_1 = 7 (5^1)+ 3 (2^1) = 7*5 + 3*2 = 41$
The base cases match.

Next, assume that for some k, the explicit formula is correct. In other words, assume that the formula $a_k = 7 (5^k)+ 3 (2^k)$ correctly calculates the same value of ak that the recursive formula did. We have to show that a k+1 is also correctly calculated. Our goal is to reach this equation: $a_{k+1} = 7 (5^{k+1})+ 3 (2^{k+1})$.
Remember: To prove the explicit formula for k+1, the tools available to us are the recursive formula for k+1, and the explicit formulas for k and k – 1.

By definition, $a_{k+1} = 7a_k - 10a_{k-1}$
Let's use the explicit formula to substitute for a_k and a_{k-1}.

$$
\begin{aligned}
a_{k+1} = 7a_k - 10a_{k-1} &= 7(7 (5^k)+ 3 (2^k)) - 10(7 (5^{k-1})+ 3 (2^{k-1})) \\
&= 49 (5^k) + 21(2^k) - 70(5^{k-1}) - 30(2^{k-1}) \\
&= 49 (5^k) + 21(2^k) - 14(5^k) - 15(2^k) \\
&= 35(5^k) + 6(2^k) \\
&= 7(5^{k+1})+ 3 (2^{k+1})
\end{aligned}
$$

We have derived P(k+1) from P(k).

144. For the following recurrence relations, derive an explicit formula for a_n.

a. $a_1 = 32$ and $a_n = 16a_{n-1}$ for $n \geq 2$.

Since this is just a homogeneous first-order recurrence relation, the answer is going to use a power of the coefficient of a_{n-1}. In other words, a power of 16.

$a_n = c\,(16^n)$

Next, we determine the coefficient by applying the base case ($n = 1$), $a_1 = 32$.
$32 = c\,(16^1) \rightarrow 16c = 32 \rightarrow c = 2$

Therefore, our answer is $a_n = 2\,(16^n)$.

b. $a_0 = 2$, $a_1 = 2$, $a_n = 4\,a_{n-1} - 4\,a_{n-2}$ for $n \geq 2$.

This is a homogeneous second-order recurrence relation. Setting the recursive formula to zero, we obtain: $a_n - 4\,a_{n-1} + 4\,a_{n-2} = 0$.

The characteristic equation is $x^2 - 4x + 4 = 0$. Factoring, we obtain $(x - 2)(x - 2) = 0$. The solution is $x = 2$, as a double root. The general form of our explicit solution is:

$a_n = c_1\,2^n + c_2\,n\,2^n$

We use both base cases to help us find the coefficients c_1 and c_2.

If $n = 0$, $a_0 = c_1\,2^0 + c_2\,0\,2^0 = 2$
This simplifies to: $c_1 = 2$.

If $n = 1$, $a_1 = c_1\,2^1 + c_2\,1\,2^1 = 2$
This simplifies to: $2c_1 + 2c_2 = 2$, or $c_1 + c_2 = 1$.

Since $c_1 = 2$, we conclude that $c_2 = -1$. Now we can write our final answer:

$a_n = 2\,(2^n) - n\,2^n$

c. $a_0 = 2$, $a_1 = 6$, and $a_n = 3a_{n-1} + 10a_{n-2}$ for $n \geq 2$.

The characteristic equation is $x^2 - 3x - 10 = 0$. Factoring, we obtain $(x - 5)(x + 2) = 0$. The solutions are 5 and -2. So, the general form of our explicit solution is:

$a_n = c_1\,5^n + c_2\,(-2)^n$

We use both base cases to determine c_1 and c_2.
If $n = 0$, $c_1\,5^0 + c_2\,(-2)^0 = 2$
This simplifies to: $c_1 + c_2 = 2$

If $n = 1$, $c_1\,5^1 + c_2\,(-2)^1 = 6$
This simplifies to: $5c_1 - 2c_2 = 6$

To solve the system of equations, multiply the first equation by 2:

$2c_1 + 2c_2 = 4$
$5c_1 - 2c_2 = 6$

Adding, we obtain $7c_1 = 10$, which means that $c_1 = 10/7$.
Substituting into the first equation $c_1 + c_2 = 2$, we find that $c_2 = 2 - 10/7 = 4/7$.

Therefore, our final answer is: $a_n = (10/7)\, 5^n + (4/7)\, (-2)^n$

d. $a_0 = 2$, $a_n = 5\, a_{n-1} + 3$ for $n \geq 1$.

This is an inhomogeneous recurrence relation. Re-write the recursive formula so that all of the sequence terms are on the left.
$a_n - 5\, a_{n-1} = 3$

The first step is to find a particular solution. The form of the particular solution depends on the form of the right side of this equation, which is just a constant. Therefore, the particular solution will be a constant as well. Call it k.

If $a_n = k$, then $a_{n-1} = k$ also.

We substitute into the recursive formula: $k = 5k + 3$. And this gives us $k = -3/4$.

Next, we seek the general solution. The characteristic equation is $x - 5 = 0$, which has the single solution $x = 5$. So, the form of the general solution, including the particular solution, is:

$a_n = c\, 5^n - 3/4$

Using the base case: If $n = 0$, $c\, 5^0 - 3/4 = 2$.
This simplifies to $c = 2 + 3/4$, or $c = 11/4$. Thus, our general solution is $11/4\, (5^n) - 3/4$.

e. $a_n = 3a_{n-1} + n$, $a_0 = 1$

This time let's use the iteration method, to see if a pattern emerges.

$$
\begin{aligned}
a_1 &= 3\, a_0 + 1 \\
a_2 &= 3\, a_1 + 2 = 3(3\, a_0 + 1) + 2 \\
a_3 &= 3\, a_2 + 3 = 3(3(3\, a_0 + 1) + 2) + 3 \\
a_4 &= 3\, a_3 + 4 = 3(3(3(3\, a_0 + 1) + 2) + 3) + 4 \\
a_5 &= 3\, a_4 + 5 = 3(3(3(3(3\, a_0 + 1) + 2) + 3) + 4) + 5 \\
&= 3^5\, a_0 + 3^4\, (1) + 3^3\, (2) + 3^2\, (3) + 3^1\, (4) + 3^0\, (5)
\end{aligned}
$$

So it looks like a_n can be written this way:
$a_n = 3^n\, a_0 + 3^{n-1}\, (1) + 3^{n-2}\, (2) + \ldots + 3^1\, (n - 1) + 3^0\, (n)$

Ignoring the first term for a moment, which has an unrelated factor of a_0, the rest of this series is almost a geometric series. Let's work it out.

Let	$S = 3^{n-1}\, (1) + 3^{n-2}\, (2) + \ldots + 3^1\, (n - 1) + 3^0\, (n)$
Then,	$S/3 = \quad 3^{n-2}\, (1) + \ldots + 3^1\, (n - 2) + 3^0\, (n - 1) + 3^{-1}\, (n)$
Subtract:	$2S/3 = 3^{n-1}\, (1) + 3^{n-2} + \ldots + 3^1 + 3^0 - 3^{-1}\, (n)$

In this last series, the terms other than the first and last are a geometric series. What is it? Let's write it in ascending powers of 3 to see:
$3^0 + 3^1 + 3^2 + \ldots + 3^{n-2}$
The common ratio is 3, the first term is 1, and there are $n - 1$ terms. The geometric series formula is the first term multiplied by $(1 - r^n) / (1 - r)$ or multiplied by $(r^n - 1) / (r - 1)$, where n is the number of terms.
In our case, $3^0 + 3^1 + 3^2 + \ldots + 3^{n-2}$ is the geometric sum equal to 1 times $(3^{n-1} - 1)/ 2$. Or in other words just $(3^{n-1} - 1)/ 2$.

So, $2S/3 = 3^{n-1} + (3^{n-1} - 1)/ 2 - n/3$. We can combine the first two terms:
$2S/3 = (3/2)(3^{n-1}) - n/3 - 1/2$
$2S/3 = (1/2)\, 3^n - n/3 - 1/2$
Multiply by 3/2 to obtain S: $S = (3/4)3^n - n/2 - 3/4$

Now, let's substitute into the formula for a_n.
$a_n = 3^n a_0 + (3/4)3^n - n/2 - 3/4$

But we know what the base case is. We were told that $a_0 = 1$. This means our formula simplifies to
$a_n = 3^n + (3/4)3^n - n/2 - 3/4$, *and:*
$a_n = (7/4)\, 3^n - n/2 - 3/4$

As an aside, it seems interesting that such an expression consisting of three fractions always evaluates to an integer. We've proved statements like this in the past. For example, here we could say that the expression $7 * 3^n - 2n - 3$ is always a multiple of 4. After doing this much work to solve such a simple looking problem, we are happy to know there is a less tedious way – to solve it like other inhomogeneous recurrence relations.

f. $a_n = 4a_{n-1} - 4a_{n-2} + 2^n$, $a_0 = 0$ and $a_1 = 2$.

This is an inhomogeneous recurrence relation. So, first we need to find a particular solution. The right side is an exponential form 2^n, so ordinarily we would expect that the form of the particular solution is $c\, 2^n$. However, we need to look at the characteristic polynomial to see how its roots turn out. In this case, we have $x^2 - 4x + 4 = 0$. This factors as $(x - 2)^2 = 0$, so we have a double root at $x = 2$. Because this root matches the base of the exponential in our particular form, and the multiplicity of this root is 2, we need to multiply $c\, 2^n$ by n^2 to give us the correct form of the particular solution, $c\, n^2\, 2^n$.

If $a_n = c\, n^2\, 2^n$, then let's figure out a_{n-1} and a_{n-2}.
$a_{n-1} = c\, (n - 1)^2\, 2^{n-1}$
$a_{n-2} = c\, (n - 2)^2\, 2^{n-2}$

Let's substitute into the given recurrence relation. For convenience, let's bring all the sequence terms to the left side.
$a_n - 4a_{n-1} + 4a_{n-2} = 2^n$
$c\, n^2\, 2^n - 4(c\, (n - 1)^2\, 2^{n-1}) + 4(c\, (n - 2)^2\, 2^{n-2}) = 2^n$
$c\, n^2\, 2^n - 2c\, (n - 1)^2\, 2^n + c\, (n - 2)^2\, 2^n = 2^n$
$c\, n^2 - 2c\, (n - 1)^2 + c\, (n - 2)^2 = 1$
$c\, (n^2 - 2\, (n - 1)^2 + (n - 2)^2) = 1$
$c\, (n^2 - 2(n^2 - 2n + 1) + (n^2 - 4n + 4)) = 1$
$c\, (n^2 - 2n^2 + 4n - 2 + n^2 - 4n + 4) = 1$
$c\, (2) = 1$

$c = ½$

So, our particular solution is $(1/2)\ n^2\ 2^n$.

Next, we find a general solution. From our characteristic roots 2 and 2, the form of the general solution is $a_n = c_1\ 2^n + c_2\ n\ 2^n. + (1/2)\ n^2\ 2^n$. Don't forget to include the particular solution here! Now, we substitute our base cases in order to solve a system of two equations in two variables.

$c_1\ 2^n + c_2\ n\ 2^n + (1/2)\ n^2\ 2^n = a_n$
If $n = 0$, $c_1\ 2^0 + c_2\ (0)\ 2^0 + (1/2)\ 0^2\ 2^0 = 0$
If $n = 1$, $c_1\ 2^1 + c_2\ (1)\ 2^1 + (1/2)\ 1^2\ 2^1 = 2$

Simplifying, this system is:
$c_1 = 0$
$2c_1 + 2c_2 + 1 = 2$
Therefore, $c_2 = 1/2$. Now our homogeneous solution is $(1/2)\ n\ 2^n$.

We add the particular and general solutions, and we are done:
$a_n = (1/2)\ n^2\ 2^n + (1/2)\ n\ 2^n$.

If desired, we can factor this answer to make it look a little nicer:
$a_n = (n(n+1)/2)\ 2^n$
$a_n = n(n+1)\ 2^{n-1}$

g. $a_n = 5a_{n-1} - 6a_{n-2} + n$, $a_0 = 19/4$ and $a_1 = 41/4$.

This is another inhomogeneous recurrence relation.
The format of the particular solution is $a_n = an + b$. Then, we should derive expressions for a_{n-1} and a_{n-2}.

$a_{n-1} = a(n - 1) + b$
$a_{n-2} = a(n - 2) + b$

Now, let's determine the values of a and b that allow $a_n = 5a_{n-1} - 6a_{n-2} + n$. Let's bring all the sequence terms to the left side:

$a_n - 5a_{n-1} + 6a_{n-2} = n$
$(an + b) - 5\ (a(n - 1) + b) + 6\ (a(n - 2) + b) = n$
$an + b - 5(an - a + b) + 6(an - 2a + b) = n$
$an + b - 5an + 5a - 5b + 6an - 12a + 6b = n$
$(a - 5a + 6a)n + (b + 5a - 5b - 12a + 6b) = 1n + 0$
$(2a)n + (-7a + 2b) = 1n + 0$

Setting the parts equal, in other words, setting the coefficients of the powers of n equal, we have this system of equations:
$2a = 1$
$-7a + 2b = 0$

Since $a = ½$, the second equation becomes $-7(1/2) + 2b = 0$
$2b = 7/2$, *so* $b = 7/4$.

This means our particular solution is $(1/2)n + 7/4$.

What is our general solution? We need to write the characteristic equation, which is:
$x^2 - 5x + 6 = 0$
This can be factored as $(x - 2)(x - 3) = 0$, *so our roots are 2 and 3.*

Thus, the general solution is $a_n = c_1 2^n + c_2 3^n + (1/2)n + 7/4$.
We use the base cases to find the values of c_1 *and* c_2.

$c_1 2^n + c_2 3^n + (1/2)n + 7/4 = a_n$
If $n = 0$, $c_1 2^0 + c_2 3^0 + (1/2)0 + 7/4 = 19/4$
If $n = 1$, $c_1 2^1 + c_2 3^1 + (1/2)1 + 7/4 = 41/4$

Simplifying, this system is:
$c_1 + c_2 + 7/4 = 19/4$ *which becomes:* $c_1 + c_2 = 3$
$2c_1 + 3c_2 + ½ + 7/4 = 41/4$ *which becomes:* $2c_1 + 3c_2 = 8$

If we multiply the first equation by –2, and add it to the second equation we obtain:
$-2c_1 - 2c_2 = -6$
$2c_1 + 3c_2 = 8$

$c_2 = 2$ *And therefore* $c_1 = 1$.

We can now write our final answer. $a_n = 2^n + (2)\, 3^n + (1/2)n + 7/4$.

h. $a_0 = 1$, $a_n = 2a_{n-1} + 3n$ for $n \geq 1$.

Another inhomogeneous recurrence relation. In the recursive rule, bring all sequence terms to the left side: $a_n - 2a_{n-1} = 3n$. *The right side is a linear function in n. So, the particular solution will have the form of* $an + b$.

If $a_n = an + b$, *then*
$a_{n-1} = a(n - 1) + b$.

Substitute into the recursive formula. Group the linear terms and constant terms separately.
$an + b - 2\,(a\,(n - 1) + b) = 3n$
$an + b - 2an + 2a - 2b = 3n + 0$
$- an + 2a - b = 3n + 0$

By setting the coefficients equal we have this system of equations for a and b:
$- a = 3$
$2a - b = 0$

The first equation gives us $a = -3$. *Substituting into the second equation, we find that* $b = -6$.
So, particular solution is $-3n - 6$.

The characteristic equation is $x - 2 = 0$. *Its solution is 2. This means the general form of our answer is*

$a_n = c\, 2^n - 3n - 6$

and we use the base case to determine the coefficient c. If $n = 0$, *we have*

$1 = c\ 2^0 - 3(0) - 6$
$1 = c - 6$
$c = 7$.

Now we are ready to write our final answer, $a_n = (7)\ 2^n - 3n - 6$.

145. Give an example of an equivalence relation (having more than one equivalence class) that can be defined on a deck of cards. Show that your relation is in fact an equivalence relation. How does your relation partition the deck of cards?

Let R say that two cards are related if they have the same color (red or black). In other words, xRy if x and y are both red or both black.

Reflexive: Is xRx true? Is every card the same color as itself? Yes!

Symmetric: If xRy, can we conclude that yRx? If x has the same color as y, then indeed y has the same color as x.

Transitive: If xRy and yRz, is xRz? If x has the same color as y, and y has the same color as z, it follows that x has the same color as z.

Since R is reflexive, symmetric and transitive, we have shown R is an equivalence relation. We have portioned the deck because a card can only have one color, and every card is either red or black.

146. Consider this relation R defined on binary strings. xRy if the last digit of x is the same as the first digit of y. Determine if the following properties hold for R: reflexive, symmetric, transitive, antisymmetric, definite, equivalence relation, partial order, total order. (Note that a "definite" relation is one in which xRy or yRx for all x and y.)

Reflexive means that for all x, the last digit of x is the same as the first digit of x. This is not true, because we can let x = 10.

Symmetric means that for all x and y, if lastDigit(x) = firstDigit(y), then lastDigit(y) = firstDigit(x). This is not true, because we can let x = 01 and y = 11.

Transitive means that if xRy and yRz, then we can conclude that xRz. This is also not true, because we can let x = 00, y = 01, and z = 11.

Antisymmetric means that if x and y are different strings and xRy, then y is not related to x. This is not true, because we can find two distinct strings that are mutually related. Let x = 01 and y = 10. The last digit of x matches the first digit of y, and the last digit of y matches the first digit of x.

R is not definite because we can find two strings x and y that have no digits in common. For example, x = 00 and y = 11. Here, lastDigit(x) = firstDigit(y) and lastDigit(y) = firstDigit(x) are both false.

For R to be an equivalence relation, it would need to be reflexive, symmetric and transitive. Well, this one isn't even close. R is not equivalence relation, in part because R is not even reflexive.

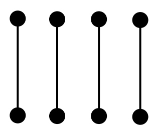
R is not a partial order because it is not reflexive to begin with.

R is not a total order because, among other things, it is not definite.

We answered "no" for all the properties!

147. Draw a simple graph that has the degree sequence indicated or explain why such a graph does not exist.

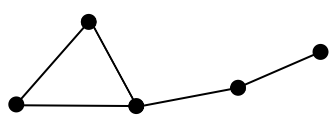
a. 1, 1, 1, 1, 1, 1, 1, 1



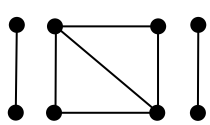
b. 1, 3, 3, 3

When we attempt to draw this graph, we find there is no way for one of the vertices to have a degree of 1. Suppose we have 4 vertices A, B, C and D. We first want A to have degree 3, so it must be adjacent to B, C and D. Next, vertex B also needs to have degree 3, so we add edges from B to C and from B to D. At this point we see the impossibility, because now both C and D must have degree at least 2.

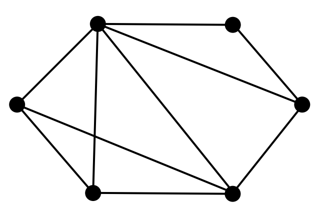
c. 1, 2, 2, 2, 3



d. 1, 1, 1, 1, 2, 2, 3, 3



e. 2, 3, 3, 3, 4, 5



f. 1, 1, 1, 1, 2, 2, 5

There is no such graph because the sum of the degrees is odd. (A quicker way to tell is to notice there is an odd number of odd degrees.)

148. Suppose a graph G has 6 vertices. How many subgraphs of G contain no edges?

If a graph has no edges, then it consists only of vertices. We choose which of the 6 vertices we want. The vertices are distinct objects. So, the question is essentially asking: how many subsets does a set of 6 elements have? The answer is 2^6.

149. If a graph is isomorphic to its complement, how many vertices should it have?

Remember that the complement graph has exactly those edges that the original graph did not. If a graph is isomorphic to its complement, then the graph has half of the maximum number of edges. If n is the number of vertices, the maximum number of edges is $n(n - 1)/2$. Thus, the graph has $n(n - 1)/4$ edges. This number must be an integer. What does it mean for the product of two numbers to be a multiple of 4? Either one of the numbers is a multiple of 4, or both numbers are even. But n and n − 1 are consecutive integers, so they cannot both be even. Therefore, either n is a multiple of 4, or n is 1 more than a multiple of 4.

150. Suppose a (simple) graph has n vertices.

a. What are the maximum and minimum number of edges the graph may have?

The maximum number of edges is $n(n - 1)/2$ and the minimum is 0.

b. If the graph has 6n edges, what does this tell us about the size of the graph (i.e. the value of n)?

$6n \le n(n - 1)/2$
$12n \le n^2 - n$
$12 \le n - 1$
$13 \le n$
So, there must be at least 13 vertices in the graph.

c. If the graph has 6n edges, how many edges are in the complement?

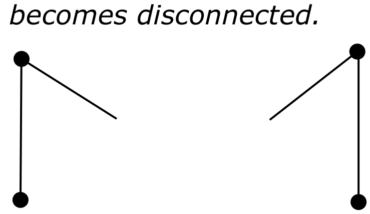
The complement has the edges that the original graph did not have. This number is the maximum minus 6n, which is $n(n - 1)/2 - 6n$.

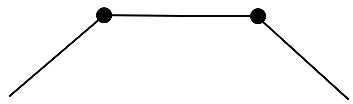
151. If a regular polygon has n vertices, how many diagonals does it have?

Picture a complete graph with n vertices. It will have $C(n, 2)$ edges. We just need to subtract the edges that form the polygon itself, and there are n of those. Answer: $C(n, 2) - n$.

152. Is the following statement true or false (and explain why) ? "If G is a cyclic graph, and G′ is obtained from G by deleting some edge, then G′ is connected."

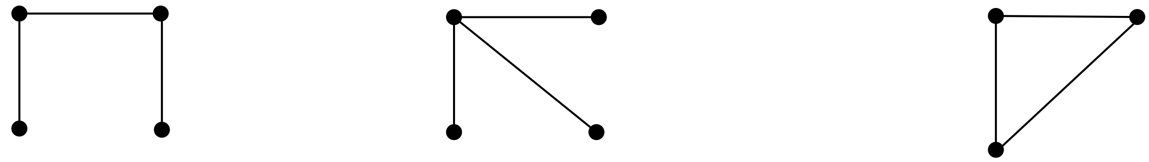
False. Here is a counterexample. This graph contains a cycle, but if we remove the edge in the middle, the graph becomes disconnected.



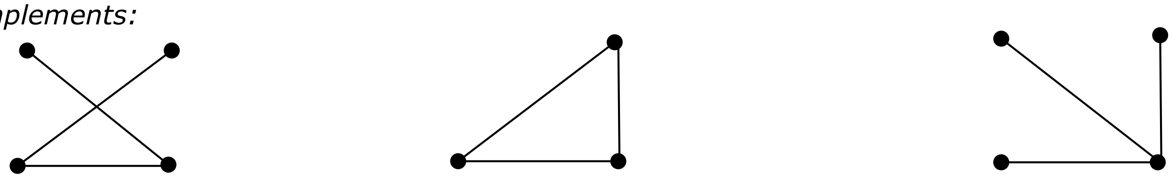


153. Draw all simple graphs having 4 vertices and 3 edges. For each graph that you draw, find its complement. What do you notice about the graphs and their complements?

Graph #1 *Graph #2* *Graph #3*



Complements:



We see that Graph #1 is its own complement. Graphs #2 and #3 are complements of each other.

154. Four couples have come to a party given by Ken and Mary. At the beginning of the evening, there is quite a lot of handshaking as people get acquainted. Of course, Ken and Mary don't shake hands with each other, and no one else shakes hands with his/her spouse. Later, Ken asks each person (including Mary) how many hands they had shaken. As it happens, no two of the answers are the same. What was Mary's answer? Hint: Draw a graph that depicts this situation. The degree of each vertex will correspond to the number of handshakes each person made.

The graph has 10 vertices, each representing one person. Because no one shakes hands with the spouse, the maximum degree is 8. A degree of 0 is possible. Ken asked nine people for the number of handshakes and received a different answer from each person. Thus, the nine vertices other than Ken's have degrees 0, 1, 2, 3, 4, 5, 6, 7, 8.

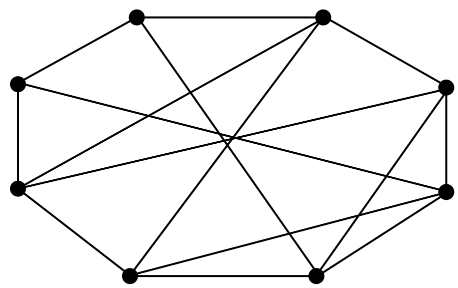
Consider the person of degree 8. This person shook hands with everyone except his or her spouse. Now, there has to be some other person in the room who shook hands with nobody. Who could that person be? It has to be the spouse of 8, because everyone else has shaken hands with 8. Thus, 8 and 0 are a couple.

Next, consider the person of degree 7. This person shook hands with everyone except his/her spouse and the person of degree 0, who is 8's spouse. We also know that someone in the room has to have degree 1. In other words, someone shook hands with exactly one person. This person of degree 1 shook hands only with 8. But we also know that everyone in the room except person 0 and 7's spouse shook hands with 7. The only person in the room to have only shaken hands with 8 and not with 7 is the spouse of 7. Thus, 7 and 1 are a couple.

We can continue with this reasoning with the other couples, and we discover that n is always married to 8 – n. But this means that some married couple is 4 and 4, a repeated number. Since Ken didn't ask himself how many times he shook hands, he must be one of the 4s. That means Mary shook hands with 4 people also.

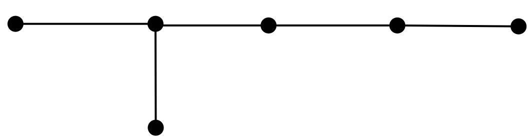
155. A Hamiltonian cycle is a cycle that visits every vertex of a graph exactly once. Draw a graph that has a Hamiltonian cycle and whose degree sequence is 3, 3, 4, 4, 4, 4, 4, 4.

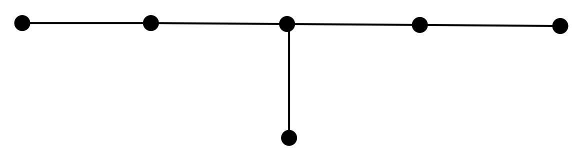
Let's draw an octagon with several of its diagonals.



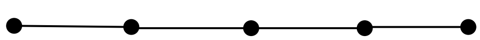
156. A "free" tree is a tree in which there is no distinct root vertex, and there is no limit on the degree of any vertex. Give an example of two free trees that have the same degree sequence but are not isomorphic.

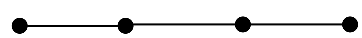
Here are two trees with degree sequence 1, 1, 1, 2, 2, 3. But they differ in which vertices are adjacent to the vertex of degree 3. In the first tree, those vertices have degrees 1 and 2. In the second tree, those vertices both have degree 2.

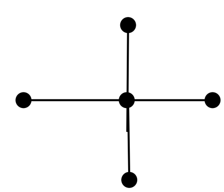




157. Draw all non-isomorphic free trees having 5 vertices. How many non-isomorphic *binary* trees with 5 vertices are there? (Note – in a binary tree, the left and right children are considered distinct.)





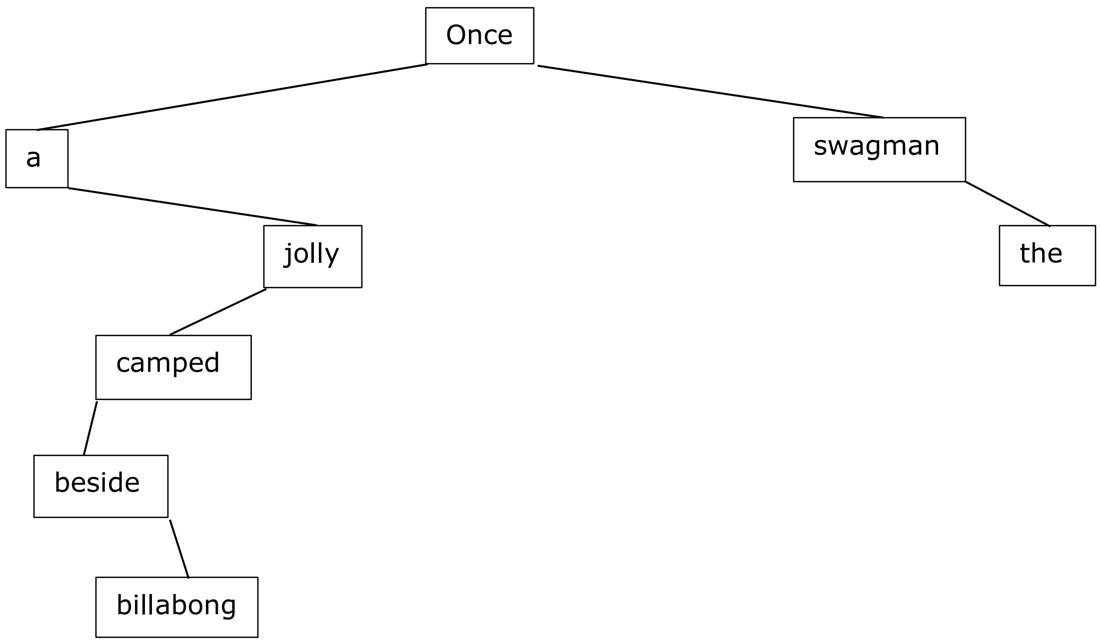


Binary trees:

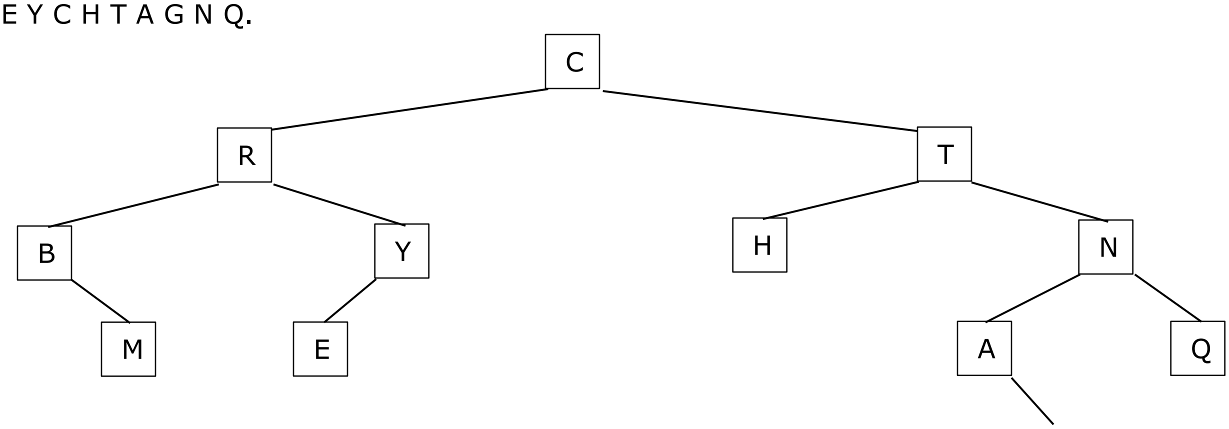
```
  16 varieties of       4 varieties of        4 varieties of        4 varieties of
    /         \            /        \            /          /            \          \
   /   thru    \          /   thru   \          / \  thru  / \          / \  thru   / \
  /             \        / \        / \        /             \         /             \
 /               \

  8 varieties of                 6 varieties of
   / \        / \                 / \        / \
  /    thru     \                / \   thru   / \
 /               \              (because there are C(4,2) positions for the grandchildren)
```

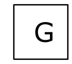
We can check our intuition by consulting the On-Line Encyclopedia of Integer Sequences, oeis.org. The first answer 3, belongs to "sequence #55", and the second answer, 42, is a Catalan number.

158. Place the following words into a binary search tree according to alphabetical order:
Once a jolly swagman camped beside the billabong



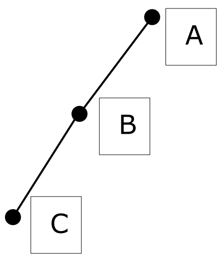
159. Draw a binary tree whose preorder traversal is C R B M Y E T H N A G Q and inorder traversal is B M R E Y C H T A G N Q.

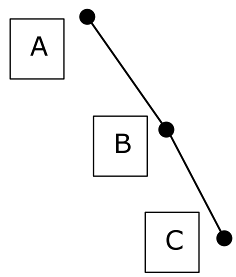




The postorder traversal is: M B E Y R H G A Q N T C.

160. Give an example of two binary trees that have matching preorder and postorder traversals, but whose inorder traversals are not the same.





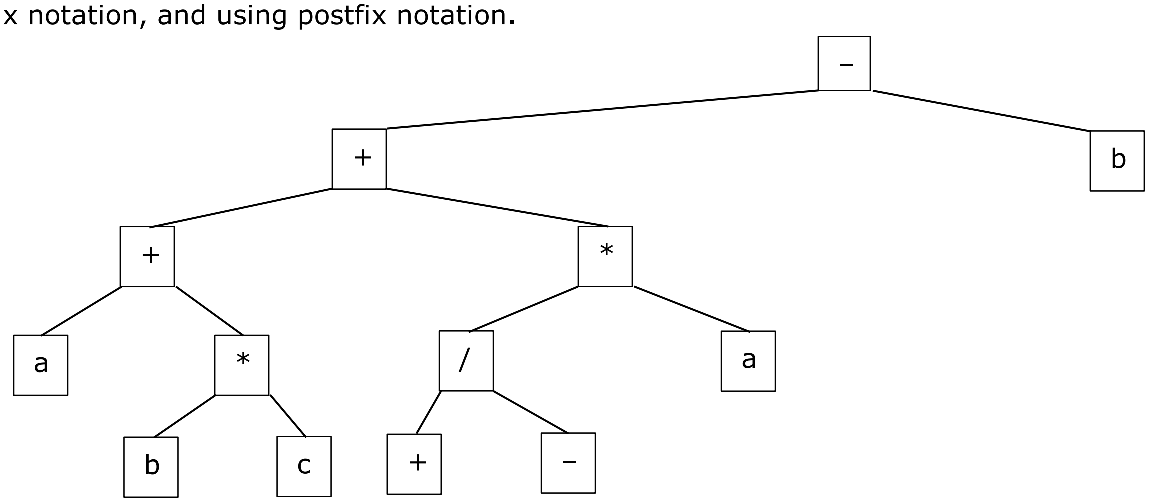
Preorder:	*ABC*	*ABC*
Inorder:	*CBA*	*ABC*
Postorder:	*CBA*	*CBA*

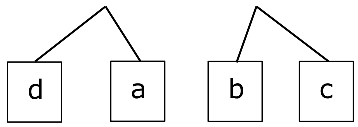
161. Practice Kruskal's and Prim's algorithms for finding a minimal spanning tree for the weighted graph given in problem 10.6.5 in the textbook. If you don't have the book, just begin by drawing a graph having 7 vertices and 10 edges, and label the edges with the weight values 1-10.

Since there are 7 vertices, any tree, including the minimum spanning tree, must have 6 edges. Kruskal adds edges of increasing weight, skipping any that would create a cycle:
AB (1), EF (2), ED (3), CD (4), FG (6), BC (7)

Prim starts with a vertex, such as A, and grows the tree along the cheapest available edge to a vertex not yet in the spanning tree, while avoiding creating a cycle:
AB (1), BC (7), CD (4), DE (3), EF (2), FG (6)

162. For the arithmetical expression a + b * c + (d – a) / (b + c) * a – b,
draw a tree that represents this expression, and then rewrite the expression in two ways: using prefix notation, and using postfix notation.





Prefix notation: – + + a * b c * / – d a + b c a b
Postfix notation: a b c * + d a – b c + / a * + b –

163. True or false:

a. There are exactly 31 subsets of the set { A, B, C, D, E }.

False. The number of subsets of a set with n elements is 2^n. If there are 5 elements, 2^5 = 32. Incidentally, the number of "proper" subsets would be $2^n - 1$. Proper subset includes sets other than the original set itself.

b. A generating function may contain an infinite number of terms.

True. A generating function may represent an infinite sequence of terms.

c. The exclusive-OR of the binary strings 1100 and 1001 is 0101.

True.
```
1100      Looking one bit at a time:  1 XOR 1 = 0
1001 XOR                              1 XOR 0 = 1
----                                  0 XOR 0 = 0
0101                                  0 XOR 1 = 1
```

d. If A = { 1, 2 } and B = { 3, 4 }, then A × B = B × A.

False. For example, A × B contains the ordered pair (1, 3). In the set B × A, every ordered pair begins with a 3 or a 4, never a 1.

e. A finite set is a set whose elements can be placed into a one-to-one correspondence with the set of positive integers.

False. The set of positive integers is infinite, so what is being described is a countable set.

f. Define a function f : R → R with f(x) = 2^x . Therefore, f is one-to-one.

True. The inverse is a function, y = $\log_2(x)$. In the inverse, every x has a unique y. Alternatively, you could show directly that f(a) = f(b) implies a = b as follows:
$2^a = 2^b$, and take the $\log_2$ of both sides to obtain a = b.

g. Define a function g: Z → Z with g(x) = x mod 100. Therefore, g is onto.

False. The range is only the integers 0..99.

h. If you flip a coin four times, the probability of achieving two heads and two tails is 1/2.

False. We choose which two outcomes are heads: C(4, 2), and then divide by the total number of coin flip outcomes, 2^4*. Answer: 6/16.*

i. A finite automaton must have at least one accept state.

False. It's possible for there to be no accept state.

j. Pascal's triangle contains every positive integer.

True. Looking at the second number of every row, we see the sequence of positive integers because it represents the expression C(n, 1) = n.

k. For all sets A and B, if $A \cup B \subseteq A \cap B$, then $A \subseteq B$.

True. If $(A \cup B)$ *is a subset of* $(A \cap B)$*, then the only elements in A and B are those that are in the intersection. In other words, A – B and B – A are empty. The statement "A is a subset of B" is true because all the elements in A are also in* $(A \cap B)$*, and every element in* $(A \cap B)$ *has to be in B.*

164. For the following questions, write a formula in terms of n.

a. What is the sum of the first n positive integers?

n(n + 1)/2

b. At a party with n people, each person shakes hands with everybody else. How many handshakes are there?

This is C(n, 2), which simplifies to: n(n – 1)/2

c. How many binary functions exist with n input variables?

There are 2^n *rows in our truth table. Each row can contain either a 0 or a 1. This means we have* 2^n *bits to write. Our answer is:*

$$2^{2^n}$$

d. How many relations exist on a set of n elements?

The adjacency matrix will have n rows and n columns. Each cell in the matrix is either 0 or 1. Thus, we have n^2 *bits to write. So, our answer is:*

$$2^{n^2}$$

e. How many functions exist, assuming that the domain and codomain have n elements? Is this number more or less than the number of relations you found in the previous part?

Each of the n elements in the domain has n choices for a functional value in the codomain. Therefore, the number of distinct functions is n^n.

The number of functions is less than the number of relations. To observe this, take the log to the base 2 of each number. Then, $\log_2$(number of relations) is n^2, and $\log_2$(number of functions) is $n \log_2 n$.