CS 261 – Summary of Counting

General ideas

1. indirect counting – Sometimes it's easier to find the number of ways something does not happen, and then subtract from the total.

2. definition of probability – (number of favorable outcomes) / (number of total outcomes)

3. In many problems you can verify your answer by trying a different approach, and seeing if you obtain the same (numerical) answer. Or, you can use the computer to do the chicken method – write a loop that literally counts the number of possibilities.

Specific Techniques

1. Making consecutive choices: xy
   The multiplication rule (a.k.a. fundamental counting principle). If there are x ways of making one choice, and y ways to make a 2nd choice, then there will be a total of xy ways to choose both.

   Typical situations: menu, license plate, or a case where some objects are being arranged into some distinct positions.

   Special cases: a general "permutation" written as P(n,r) means the number of ways r objects are selected from n and then permuted (without repetition). This is almost like the factorial situation, but we don't use up all n objects.

2. Factorials: n!
   To arrange n objects in a row, we can do it in n! ways.

   Typical situations: people sitting in a row of chairs, books on a shelf

   Special cases: (n – 1)! for arranging objects in a circle (like a dinner table), and
   (n – 1)!/2 if the circle can be turned upside-down (as in a key chain)

   Also watch out for cases where we want to separately sort different kinds of books, people (like men, women, boys, girls). You may need to break up into cases when given a restriction like certain things/people must be in certain places (e.g. all on one side or alternating positions).

3. Distinguishable permutations: n! / (a!b!…)
   If we want to arrange n objects in a row, but a of them are identical to one other, and b of them are also identical, and so on.

   Typical situations: letters of a word

   Special cases: If the groups *themselves* are indistinguishable, then we need another factorial in the denominator.

4. Combinations: C(n, r)
The notation for combinations varies. We tend to use the big parentheses notation, but for ease of typing, I will use C(n, r) here. To choose r things from a set of n, you can do it in C(n, r) ways. Note that when we do combinations, order usually does not matter unlike permutations. The r objects do not have a "pecking order" like 1^{st}, 2^{nd}, 3^{rd}. They are simply being chosen.

Typical situations: committee, subset, playing cards

Notes:

a. Beware of double counting in cases of "at least one".

b. Don't unnecessarily impose an order on what is being chosen. If you want 2 cards from the deck, say C(52, 2), not C(52, 1) * C(51, 1).

c. Many times, a combination question can be thought of as a case of distinguishable permutations (section 3 above). So likewise note that we sometimes have a situation where the groups/teams we are forming are indistinguishable, and we need to divide by a factorial so we don't unnecessarily impose an order on the groups.

d. When using indirect counting, the "total" is 2^n if you are referring to all possible ways of choosing any number r from the set of n. This is because the sum of the numbers on row n of Pascal's triangle is 2^n.

e. Choosing r out of n is basically the same as *not choosing* n – r.
Thus, C(n, r) = C(n, (n – r)).

f. Often we need to multiply different combination expressions because we are making consecutive choices (like section 1 above). And whenever we say "and", we multiply. For example, we could be choosing men and women, different kinds of cards for a poker hand, good and defective machines, etc.

5. "Ball in urn": C(n + r – 1, n)
This is the number of ways that n identical objects can be placed into r categories. Note that the formula could also be written C(n + r – 1, r – 1). Basically, the "r – 1" comes from the fact that r categories can be divided with r – 1 dividers. We can model the situation by writing 0's for the identical objects and 1's for the dividers. Now we are looking at a binary number having n + r – 1 digits, and all we have to do is figure out where to put the 1's, which is just a combination formula. Anyway, this is just where the formula comes from…

Note that a common tell-tale sign that you are dealing with a "ball in urn" question is that you are not given the sizes of the categories, because this is actually the point of the question: how many of each can I choose?

Typical situations:
placing n identical balls into r urns
n identical objects to distribute among r people
need to buy many items of different kinds – want to know how many of each
number of non-negative integer solutions to an equation like a + b + c = 12

Notes:
a. Numbers appearing in this question may refer to different values of n – we may need to repeat the formula several times, as in the "coin collector" problem.

b. Beware of the possibility of running out of a particular category. (See section 6 on set intersection also.)

c. There is a special case of needing "at least" k or some number in each category. This reduces the number of identical objects that can freely go into whatever category. Also, different categories may have different minimum requirements. And in an abstract situation like an integer equation, the minimum may even be negative.

d. Special case where if you are not required to use up all n objects (if they don't have to all be distributed), then we can, in effect, create a new category # (r+1) to take up all these unused objects. See the $20,000 investor question from the supplement.

6. Inclusion/exclusion – counting how many items belong in a set union or intersection. We have slightly different formulas based on how many sets we are dealing with, but they all follow the same pattern:

Union of 2 sets: A + B – AB
Union of 3 sets: A + B + C – AB – AC – BC + ABC
Union of 4 sets: A + B + C + D – (2-set intersections) + (3-set intersections) – ABCD

This is the "true" way to handle an *or* situation. For example, the number of ways to choose a jack or a club from a deck of cards is 4 + 13 – 1 (the –1 represents the jack of clubs, which we don't want to count twice).

Let's try some contrasting examples.

1. We want to buy 10 cans of soup, but the store only has 3 varieties available. How many ways can I choose how many of each kind of soup to buy?

> This is a "ball in urn" question, where we have 10 identical objects and 3 categories to put them, so n = 10 and r = 3. The answer is C(10 + 3 – 1, 10) = C(12, 10) or C(12, 2).

2. 10 *people* want to eat some soup. In my cupboard there are 3 kinds of soup to be found: tomato, chicken noodle and alphabet. How many ways can the 10 people choose their soup? (Assume each person has just 1 soup!)

> We ask each person which kind of soup he wants, and we expect 1 of 3 answers from each person. This is the multiplication rule: 3^{10}.

3. 10 people want soup, and all I have in the cupboard are 5 cans of tomato, 3 of chicken noodle and 2 alphabet. How many ways can each person have soup?

> This is a combination question. We choose which 5 of the 10 people eat tomato, which 3 of the remaining 5 will have chicken noodle, and we'll then have no choice about the 2 people who get alphabet. The answer is C(10, 5) * C(5, 3) * C(2, 2).

4. We want to buy 10 cans of soup, but the store is nearly sold out and only has on the shelf 5 cans of tomato, 3 of chicken noodle and 2 alphabet. How many ways can we choose how many of each to buy?

> Not much going on in this question. The question is just asking to repeat the given information. We really have no choice but to buy all the soup available. So the answer is 1.

What do the chicken methods look like for the above questions?

1.
```
for i = 0 to 10
    for j = 0 to 10
        for k = 0 to 10
            if sum == 10
                increment count
```

2. Ten nested loops where each variable goes from 1 to 3.

3. Ten nested loops, and inside the innermost we test to see if 5 of the variables equal 1, 3 of the variables equal 2, and 2 of the variables equal 3.

4.
```
for i = 0 to 10
    for j = 0 to 10
        for k = 0 to 10
            if i == 5, j == 3 and k == 2
                this is it!  (Obviously this happens only once.)
```

These last 2 questions would be solved somewhat differently if we had more than just 10 cans of soup to begin with. These are more complicated situations.

5. 10 people want soup, and all I have in the cupboard are 6 cans of tomato, 7 of chicken noodle and 8 alphabet. How many ways can each person have soup?

> There may be an easier way to do this, but as I'm typing this solution, the only way that comes to mind is to enumerate the cases that add up to 10 cans that we need, and consider each case as a simple combination question. For example, one case is where we want 6 tomato, 4 chicken noodle, and 0 alphabet. Then we choose which 6 of 10 people eat tomato, and the other 4 people will automatically eat chicken noodle: C(10, 6) * C(4, 4). But unfortunately this question is quite tedious because there are many ways the 6, 7 and 8 cans can make partial sums to 10. They are:
>
> 6+4+0 6+3+1 6+2+2 6+1+3 6+0+4
> 5+5+0 5+4+1 5+3+2 5+2+3 5+1+4 5+0+5
> 4+6+0 4+5+1 4+4+2 4+3+3 4+2+4 4+1+5 4+0+6
> etc.
>
> And we have to be careful about avoiding impossible cases like 7+3+0 because there are only 6 cans of tomato soup. Maybe there is a simpler way I can't think of, but for now, we'll file this under yucky questions.

One motivation for solving a question like this is to scale down the numbers in the question, so that you can study all the cases carefully and even try the chicken method to see if your general intuitive solution works.

6. We want to buy 10 cans of soup, but the store only has available 6 tomato, 7 chicken noodle and 8 alphabet. How many ways can we choose the number of each to buy?

If there were no limitations, it would have been n = 10 and r = 3: C(10+3–1, 10).

To solve this problem, we use the indirect method. We start with the total number of ways, and subtract impossible cases. The impossible cases are those where we would have had 7+ tomato, 8+ chicken noodle or 9+ alphabet. Calculate each of these separately and subtract from the total.

7+ tomato means we freely choose 3: n = 3, r = 3: C(5, 3)
8+ chicken noodle means we freely choose 2: n = 3, r = 2: C(4, 2)
9+ alphabet means we freely choose 1: n = 3, r = 3: C(3, 1).

So our answer is C(12, 10) – C(5, 3) – C(4, 2) – C(3, 1).

There would be an extra wrinkle if we had to handle intersections. Note that in question 6, the three cases are mutually exclusive. If you use up all the tomato soup, you won't run out of the rest. Let's make the limitations more interesting.

7. We want to buy 10 cans of soup, but the store only has available 3 tomato, 4 chicken noodle and 5 alphabet. How many ways can we choose the number of each to buy?

The total with no restrictions would be the same: C(10 + 3 – 1, 10) = C(12, 10).

4+ tomato means we freely choose 6: n = 6, r = 3: C(8, 6)
5+ chicken noodle means we freely choose 5: n = 5, r = 3: C(7, 5)
6+ alphabet means we freely choose 4: n = 4, r = 3: C(6, 4)

But, what about the situation of choosing 4+ tomato and 5+ chicken noodle? And also 4+ tomato and 6+ alphabet? We have to add those cases back in.

4+ tomato and 5+ c.n.: freely choose 1: n = 1, r = 3: C(3, 1)
4+ tomato and 6+ alpha: freely choose 0: n = 0, r = 3: C(2, 0)

Answer = C(12, 10) – C(8, 6) – C(7, 5) – C(6, 4) + C(3, 1) + C(2, 0).

We could verify our answer with the chicken method:

```
for i = 0 to 3
   for j = 0 to 4
      for k = 0 to 5
         if the sum == 10
            increment count
```

Good news! Both answers come out to 6.

CS 261 – Practice with Permutations and Combinations

1. The menu of a small restaurant offers 6 appetizers, 8 entrees and 3 desserts.

   (a) In how many ways can we order a meal (one appetizer, one entrée, and one dessert)?

   (b) How would your answer change if the appetizer and dessert were optional?

2. From a group of 3 freshmen, 4 sophomores, 5 juniors and 2 seniors, a committee is to be formed taking one person from each class. In how many ways could this committee be chosen?

3. How many different 7-place license plates can be made if there are 3 letters followed by 4 numbers if:

   (a) there are no restrictions?

   (b) repetitions are prohibited?

4. A delicatessen has ten customers: six women and four men. Each takes a number to be served.

   (a) In how many ways could the customers draw the numbers?

   (b) In how many ways could the numbers be drawn if all the men are to be served consecutively?

   (c) In how many ways could the numbers be drawn if all the men are to be served consecutively, and all the women are to be served consecutively?

5. Mr. Jones has 10 books on a shelf: 4 math, 3 chemistry, 2 history and 1 psychology. How many arrangements are possible if books of the same subject are together?

6. How many possible arrangements of the letters of PEPPER are there?

7. An Olympic skiing event has 10 competitors: 4 American, 3 Norwegian, 2 Austrian and 1 Canadian. If the results of the competition list the only the nationalities of the 10 skiers, how many rankings are possible?

8. A committee of three is formed from a larger group of 20. How many ways could this be done?

9. How many subsets of the set { e, i, g, h, t } contain:

   (a) no vowels

   (b) exactly one vowel

   (c) both vowels

10. A police department has ten officers. Policy is to have 5 on the streets, 2 in the office, and 3 on reserve. How many possible ways are there to divide the force?

11. In how many ways can 10 balls be placed in 4 urns if:

    (a) the balls are distinct?

    (b) the balls are identical?

12. How many distinct ordered pairs/triples of non-negative integers satisfy:

    (a) $x + y = 3$?

    (b) $x + y + z = 7$?

    (c) Redo the previous part, assuming that $x \geq 1$, $y \geq 2$, and $z \geq -1$.

13. An investor has $20,000 and may choose among four investments. Each investment permits deposits only in thousand-dollar increments. How many investment strategies are possible if:

    (a) all $20,000 is to be invested?

    (b) the investor may choose not to invest all the money?

14. A student is to answer 7 questions out of 10 on an examination. How many choices are there if he must answer at least 3 out of the first 5?

15. A group of 5 men, 5 women, 17 boys and 18 girls are buying tickets to a hockey game. Assume that they will all sit in the same row of the arena. In how many ways can they arrange themselves if all the men sit together, all the women sit together, all the boys sit together, and all the girls sit together?

16. In how many ways can four bridge hands (13 cards) be dealt?

17. If twelve people are to be divided into groups of 3, 4 and 5, then how many ways can this be done?

18. In how many ways is it possible to distribute 7 gifts among 3 children if one child receives 3 gifts, and the other two each receive 2?

19. The Pickens County School Board oversees a district of 24 schools. If 100 brand new blackboards are acquired over the summer, in how many ways can they be distributed and installed among the various schools if:

    (a) there are no restrictions?

    (b) each school must get at least one blackboard?

20. An elevator starts at the basement with 8 people, and stops at floors 1-6 in succession. Assume that no one else gets on the elevator. In how many ways can the elevator empty itself if:

    (a) the people are distinct?

    (b) the people are not considered distinct?

21. A coin auction is selling 4 Maple Leaves, 5 Double Eagles and 6 Krugerrands. Coins of the same type are identical. In the audience are 5 bidding collectors. In how many ways can the coins be distributed among the collectors?

22. A trade conference involving the heads of state of 10 European countries is taking place. In a publicity photo, they are to be shown sitting in a single row, smiling. In how many ways can they be seated if the prime ministers of the U.K. and Italy do not sit together, while the French and Russian presidents do sit together?

23. The answer is in the computer, if only you knew where to look… For this problem let's assume that your computer's hard drive is 8 GB, and let's say all this memory just represents a lot of numbers. To be precise, let's say that we have 2^{30} numbers, and each one has a value between 0 and $2^{32} - 1$. What is the probability that some arbitrary number like 42 is contained in any of these 2^{30} memory locations?

24. What is the probability of rolling a "Yahtzee"? Assume the following strategy: You have 5 dice, and the goal is to get the same number on each die. You get three chances. After each of the first 2 rolls, you keep (don't roll) the most common number you have. For example, if the first roll gives you 1, 1, 2, 4, 5, then you keep the ones and roll the other 3 dice. Then, if the second roll yields 1, 3, 6, then you keep the 1 so that now you have three ones. On the third roll you roll just the 2 remaining dice in the hopes they both come up 1.

    Be careful: This can get tricky. Here's another example. Let's say that in the situation we had above, the second roll yielded three 2's. So now you have two 1's from the first roll and now three 2's. For the third roll, you would abandon the 1's since you now have more 2's, since you are more likely to win rolling just two dice instead of three.

25. Suppose you run an opera company that is planning on performing 4 operas during the month of April. Your task is to schedule the operas – that is, decide which operas will be performed on what days. The 4 operas taken together comprise a single coherent story, so they must be performed in a specified order. You may assume that each opera will be performed only once. However, each opera is so long that only one can be scheduled on any given day. Thus, the problem boils down to figuring out which days in April you want the company to perform. How many ways can this be done? Determine the answer for each of the following situations. Solve each part independently of the others. (Note – if you get any of these wrong, your dragon prop will be sent to Beirut!)

    (a) There are no further stipulations.

    (b) You are allowed to perform up to 2 operas per day.

    (c) One of the props for the 3^{rd} opera won't be available until the 17^{th}.

    (d) The 2^{nd} opera can only be performed on the 7^{th}, 14^{th} or 21^{st}.

    (e) For the sake of continuity, once the opera series has started, you cannot have more than 6 consecutive days without another performance (of the next opera in the series). For example, if the first opera is on the 3^{rd}, the latest you may schedule the 2^{nd} opera will be the 10^{th}.

    (f) All the operas are to be performed on the same day of the week.

26. A palindrome is a word, or sequence of symbols, that reads the same backwards as forwards. For example, "level" and "ghhg" are palindromes.

    (a) How many seven-letter palindromes are there based on our 26-letter alphabet?

    (b) How many eight-letter palindromes?

    (c) Write a general expression for the number of palindromes in terms of the length n.

27. How many 4-digit numbers…

    (a) Do not contain a 7?

    (b) Are odd and have no repeated digits?

28. At a world speed skating championship, there are 8 contestants, each from a different country. Assume that everyone's time is distinct (no ties).

    (a) How many ways can the gold, silver and bronze medals be awarded?

    (b) How many ways can the medals be awarded if the Dutch sprinter must win a medal, and the Belgian competitor is disqualified?

29. Suppose you flip a coin 8 times. What is the probability that you get 4 heads and 4 tails?

30. Consider the set of positive integers from 1 to 7700, inclusive. How many of these numbers are divisible by either 7 or 11?

31. Consider 3 letter strings of capital letters, AAA to ZZZ.

    a. In how many of these strings do we see two consecutive letters that are the same?

    b. How many strings do not have any two consecutive letters that are the same?

    c. In how many strings do we have three distinct letters?

    d. How many strings have at least one occurrence of a repeated letter?

32. How many bit strings of length 16 contain

    a. at most three 1's?

    b. at least seven 1's and at least seven 0's?

33. Our alphabet has 21 consonants and 5 vowels. Consider the set of all 9-letter strings of capital letters. How many contain 5 consonants and 4 vowels?

34. Consider the set of 5-letter strings of capital letters. How many contain both an A and a B?

35. How many ways are there to travel from the point (0, 0, 0) to (5, 7, 3) by taking integer steps in the positive x direction, the positive y direction and the positive z direction?

36. How many 6-digit numbers have a sum of digits of 13?

37. If a machine has a failure rate of 1/x in one minute, then how many minutes must pass before the failure rate reaches 1/2? Express your answer in terms of x. Then, evaluate your answer when x = 1 million, and again when x = 2 million.

38. In how many ways can we permute the letters ABCDEFG, where A must appear to the left of B, and B must appear to the left of C?

39. In how many ways can we deal a poker hand that contains

    a. at least 2 aces?

    b. two or three face cards?

40. Suppose the National Hurricane Center is tracking three tropical waves in the Atlantic Ocean. The probability that each will develop into a tropical depression over the next 5 days is 10%, 40% and 30%, respectively. Think about the probability that at least one will develop.

    a. Why is the answer not 10% + 40% + 30%?

    b. What is the correct probability?

41. A drawer contains 2 blue, 4 red, and 2 black socks. If 2 socks are randomly taken out of the drawer, what is the probability that they will be of the same color?

42. Suppose a sports league has N teams. If each team is to play every other team exactly once, then how many games must be scheduled?

43. A small company has 15 employees, comprising 9 men and 6 women. A hiring committee is to be formed of 2 men and 2 women. This committee may not include the CEO, who is one of the 9 men. Therefore, in how many ways can the members of the committee be selected?

44. Suppose that at a certain college, 90% of the graduates earn a bachelor's degree, and 10% earn a master's degree. The percentage of graduating students who have relevant job experience is 50% among those with a bachelor's degree and 90% for those with a master's degree.
    a. What percentage of those with job experience have earned a master's degree?

    b. What percentage of those without job experience have earned a master's degree?

45. Suppose 4 men and 4 women are sitting in a row.

    a. In how many ways could this be done?

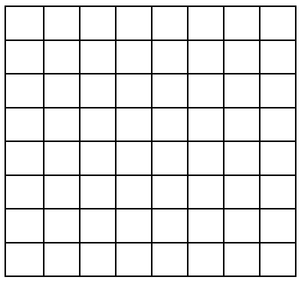
    b. In how many ways can they sit so that that there is at least one man between each pair of women?

    c. What if we had 8 men instead of 4?

46. The World Series is a best-of-7 series. The winner of the series is the team that first wins 4 games. Once a team has won 4 games, the series is over, even if 7 games have not all been played. In this question, assume that all games are played on a neutral field, and the probability of winning any single game is 1/2.
    a. If a team loses the first two games of the World Series, what is the probability that it will win the series overall?

    b. Knowing that your team won the World Series, what is the probability that it lost the first two games?

47. In the following figure resembling a chessboard,



a. How many squares do you see?

b. How many rectangles do you see?