INDUCTION EXAMPLES (sums)

P_n says that for all $n \geq 1$:

$$1^3 + 2^3 + 3^3 + \dots + n^3 = \left[\frac{n(n+1)}{2}\right]^2$$

First we show P_1 is true, P_1 says

$$1^3 = \left[\frac{1(1+1)}{2}\right]^2$$

$$1 = \left(\frac{2}{2}\right)^2 \text{ which is true.}$$

Now assume P_k is true for some $k \geq 1$. In other words, assume:

$$1^3 + 2^3 + 3^3 + \dots + k^3 = \left[\frac{k(k+1)}{2}\right]^2$$

add next term to both sides:

$$+ (k+1)^3 \qquad + (k+1)^3$$

now we have: $1^3 + 2^3 + 3^3 + \dots + k^3 + \boxed{(k+1)^3} = \left[\frac{k(k+1)}{2}\right]^2 + \boxed{(k+1)^3}$

$$= \frac{1}{4}k^2(k+1)^2 + (k+1)^3$$

$$= \frac{1}{4}(k+1)^2\left[k^2 + 4(k+1)\right]$$

$$= \frac{1}{4}(k+1)^2(k^2 + 4k + 4)$$

$$= \frac{1}{4}(k+1)^2(k+2)^2$$

$$= \left[\frac{(k+1)(k+2)}{2}\right]^2$$

This note is not part of the proof, but remember our goal is for the right side to look like P_{k+1}:
In other words, plug $k+1$ in for n:
$$\left[\frac{n(n+1)}{2}\right]^2 = \left[\frac{(k+1)(k+2)}{2}\right]^2$$

and this last equation is P_{k+1}.

Since P_1 is true and P_k implies P_{k+1}, P_n is true for all $n \geq 1$.

P_n says that for all $n \geq 1$:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{n(n+1)} = \frac{n}{n+1}$$

P_1 says that:

$$\frac{1}{1(1+1)} = \frac{1}{1+1}$$

$$\frac{1}{2} = \frac{1}{2}$$ which is true.

Now assume P_K is true. In other words, assume that for some $k \geq 1$:

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)} = \frac{k}{k+1}$$

we add next term to both sides to obtain

$$\frac{1}{1\cdot 2} + \frac{1}{2\cdot 3} + \frac{1}{3\cdot 4} + \cdots + \frac{1}{k(k+1)} + \boxed{\frac{1}{(k+1)(k+2)}} = \frac{k}{k+1} + \boxed{\frac{1}{(k+1)(k+2)}}$$

$$= \frac{k(k+2)}{(k+1)(k+2)} + \frac{1}{(k+1)(k+2)}$$

$$= \frac{k(k+2)+1}{(k+1)(k+2)}$$

$$= \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)}$$

$$= \frac{k+1}{k+2}$$

Scratch work, not part of proof...
Our goal is for right side to look like $\frac{n}{n+1}$ when $n = k+1$
which is $\frac{k+1}{k+2}$.

This last equation is P_{K+1}.

Since P_1 is true and P_K implies P_{K+1},
P_n is true for all $n \geq 1$.

P_n says that for all $n \geq 0$:

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \ldots + (n+1)2^{n+1} = n \cdot 2^{n+2} + 2.$$

The base case is $n = 0$.

P_0 says: $\quad (0+1)2^{0+1} = 0 \cdot 2^{0+2} + 2$

$2^1 = 2$, which is true.

Now assume P_K is true. That is, assume that for some integer $k \geq 0$:

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \ldots + (k+1)2^{k+1} = k \cdot 2^{k+2} + 2$$

Add the next term to both sides:

$$1 \cdot 2^1 + 2 \cdot 2^2 + 3 \cdot 2^3 + \ldots + (k+1)2^{k+1} + \boxed{(k+2)2^{k+2}} = k \cdot 2^{k+2} + 2 + \boxed{(k+2)2^{k+2}}$$

$$= \underbrace{k \cdot 2^{k+2} + k \cdot 2^{k+2}}_{} + 2 \cdot 2^{k+2} + 2$$

$$= k\left(2^{k+2} + 2^{k+2}\right) + 2^1 \cdot 2^{k+2} + 2$$

$$= \underbrace{k \cdot 2^{k+3} + 2^{k+3}}_{} + 2$$

$$= (k+1)\,2^{k+3} + 2$$

Scratch work, not part of proof.
Our goal is for right side to look like $n \cdot 2^{n+2} + 2$ when $n = k+1$ which is:
$(k+1)\,2^{k+3} + 2$

This last equation is P_{K+1}.

Since P_0 is true and P_K implies P_{K+1}

P_n is true for all $n \geq 0$.

Induction – in-class examples

A divisibility problem: Show that for all (positive integers) $n \geq 1$, $3 \mid (4^n - 1)$.

Step 1: $P(1)$ says $3 \mid (4^1 - 1)$
$3 \mid 3$ which is true.

Step 2: Assume $P(k)$ is true for some $k \geq 1$.

In other words, assume $3 \mid (4^k - 1)$.

Our goal is to show that $3 \mid (4^{k+1} - 1)$.

Consider the difference between these 2 numbers:

$$
\begin{aligned}
(4^{k+1} - 1) - (4^k - 1) &= 4^{k+1} - 4^k - 1 + 1 \\
&= 4 \cdot 4^k - 4^k \\
&= 3 \cdot \underbrace{4^k}_{\text{integer}}
\end{aligned}
$$

Since $3 \mid (4^k - 1)$ and
$3 \mid [(4^{k+1} - 1) - (4^k - 1)]$

then 3 divides their sum, which is $4^{k+1} - 1$

In other words, $3 \mid (4^{k+1} - 1)$, which is $P(k+1)$.

Showing an inequality is true —
For all (positive integers) $n \geq 3$, $2n+1 < 2^n$.

Step 1: $P(3)$ says $2(3)+1 < 2^3$
$7 < 8$ which is true.

Step 2: Assume $P(k)$ is true for some $k \geq 3$.
In other words, assume $2k+1 < 2^k$. ①

Our goal is to show that $2k+3 < 2^{k+1}$.

Consider the difference between these 2 inequalities:

consider $(2k+3)-(2k+1) < 2^{k+1} - 2^k$

$$2 < 2^{k+1} - 2^k$$
$$2 < 2 \cdot 2^k - 2^k$$
$$2^1 < 2^k$$

Since $k \geq 3$, we can definitely say $2^1 < 2^k$, and this is equivalent to
$(2k+3)-(2k+1) < 2^{k+1} - 2^k$ ②

Now that we know inequalities ① and ② are true, we can add them to obtain

$$2k+3 < 2^{k+1}$$

which is $P(k+1)$.

Show that for all $n \geq 0$, $2^n < (n+2)!$

"Base case" — $P(0)$ says that $2^0 < (0+2)!$

$1 < 2$ which is true.

"Inductive step" — Assume $P(k)$ is true for some $k \geq 0$.

In other words, assume $2^k < (k+2)!$ ①

We need to show that $2^{k+1} < (k+3)!$

Consider the quotient of these 2 inequalities:

consider $$\frac{2^{k+1}}{2^k} < \frac{(k+3)!}{(k+2)!}$$

$$2 < k+3$$

$$-1 < k$$

Since $k \geq 0$, we can definitely say $-1 < k$, and this is equivalent to $\frac{2^{k+1}}{2^k} < \frac{(k+3)!}{(k+2)!}$ ②

Since inequalities ① and ② are true (and all represent positive numbers), we can multiply them to obtain —

$$2^{k+1} < (k+3)!$$

which is $P(k+1)$.

For all $n \geq 2$, $\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \dots + \frac{n}{n+1} < \frac{n^2}{n+1}$

"Base case" is $P(2)$ which says

$$\frac{1}{2} + \frac{2}{3} < \frac{2^2}{2+1}$$

$$\frac{7}{6} < \frac{4}{3}$$ which is true.

"Inductive step" — Assume $P(k)$ is true for some $k \geq 2$.

in other words assume

$$\frac{1}{2} + \frac{2}{3} + \dots + \frac{k}{k+1} < \frac{k^2}{k+1} \qquad (1)$$

We need to show $P(k+1)$ is therefore true :

$$\frac{1}{2} + \frac{2}{3} + \dots + \frac{k}{k+1} + \frac{k+1}{k+2} < \frac{(k+1)^2}{k+2}$$

Consider the difference between these 2 inequalities, which is

$$\frac{k+1}{k+2} < \frac{(k+1)^2}{k+2} - \frac{k^2}{k+1} \qquad (2)$$

(Since $k \geq 2$, all denoms positive)

$$(k+1)(k+1) < (k+1)^3 - k^2(k+2)$$

$$k^2 + 2k + 1 < k^3 + 3k^2 + 3k + 1 - (k^3 + 2k^2)$$

$$k^2 + 2k + 1 < k^2 + 3k + 1$$

$$0 < k$$

Since $k \geq 2$, we can definitely say $0 < k$, and this is equivalent to (2).

Since (1) and (2) are true, we can add them to obtain

$$\frac{1}{2} + \frac{2}{3} + \dots + \frac{k+1}{k+2} < \frac{(k+1)^2}{k+2}$$

which is $P(k+1)$.

MORE EXAMPLES OF INDUCTION (inequalities)

4.3.17

P_n is the statement $2^n < (n+2)!$ $\quad n \geq 0.$

P_0 says that
$$2^0 < (0+2)!$$
$$1 < 2!$$
$$1 < 2 \quad \text{which is true.}$$

Assume P_k is true. That is, assume that

(1) $\quad 2^k < (k+2)!$ for some $k \geq 0.$

Since $k \geq 0$ we can also say:
$$-1 < k$$
$$2 < n+3$$
$$2 \cdot \frac{2^k}{2^k} < (n+3)\,\frac{(n+2)!}{(n+2)!}$$

because we can multiply both sides by one.
Now we have

(2) $$\frac{2^{k+1}}{2^k} < \frac{(k+3)!}{(k+2)!}.$$

Next, we combine inequalities (1) and (2) using the basic rule "$a<b$ and $c<d$ implies $ac<bd$" to obtain:
$$2^k \cdot \frac{2^{k+1}}{2^k} < (k+2)!\,\frac{(k+3)!}{(k+2)!}$$
$$2^{k+1} < (k+3)!$$

which is P_{k+1}.

Scratch work, not part of the proof.

We want to use the rule that
$a < b$ and $c < d$
implies
$ac < bd$
where:
$a = f(k)$ $\quad c = \dfrac{f(k+1)}{f(k)}$
$b = g(k)$ $\quad d = \dfrac{g(k+1)}{g(k)}$

Let's see if in fact $c<d$.
$\dfrac{f(k+1)}{f(k)}$ vs. $\dfrac{g(k+1)}{g(k)}$ means
$\dfrac{2^{k+1}}{2^k}$ vs. $\dfrac{(k+3)!}{(k+2)!}$

2 vs. $k+3$
-1 vs. k
The relationship is $<$ since k is already ≥ 0.

Since P_0 is true and P_k implies P_{k+1},
P_n is true for all $n \geq 0$.

4.3.18

P_n is the statement

$$\sqrt{n} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} \qquad \text{for } n \geq 2.$$

P_2 says that

$$\sqrt{2} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}}$$

$$(\approx 1.4) < (\approx 1.7) \quad \text{which is true.}$$

Assume P_K is true, for some $k \geq 2$:

(1) $$\sqrt{K} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{K}}$$

Note that since k is positive, we can say that $\sqrt{\frac{k}{k+1}}$ is also positive, so that

$$0 < \sqrt{\frac{k}{k+1}}$$

$$1 < 1 + \sqrt{\frac{k}{k+1}}$$

which, by the calculations in the sidebar is equivalent to saying

(2) $$\sqrt{K+1} - \sqrt{K} < \frac{1}{\sqrt{K+1}}$$

Now we can combine inequalities (1) and (2) by adding them together:

$$\sqrt{K} + \sqrt{K+1} - \sqrt{K} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{K}} + \frac{1}{\sqrt{K+1}}$$

$$\sqrt{K+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{K+1}}$$

and this is P_{K+1}.

Scratch work:

P_K says

$$\sqrt{K} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{K}}$$

P_{K+1} says

$$\sqrt{K+1} < \frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{K}} + \frac{1}{\sqrt{K+1}}$$

So let's consider

$$f(k+1) - f(k) \ \underline{vs.}\ g(k+1) - g(k)$$

$$\sqrt{K+1} - \sqrt{K} \ \underline{vs.}\ \frac{1}{\sqrt{K+1}}$$

multiply both sides by $\sqrt{K+1} + \sqrt{K}$

$$(\sqrt{K+1} + \sqrt{K})(\sqrt{K+1} - \sqrt{K}) \ \underline{vs.}\ \frac{\sqrt{K+1} + \sqrt{K}}{\sqrt{K+1}}$$

$$K+1-K \quad vs. \quad 1 + \frac{\sqrt{K}}{\sqrt{K+1}}$$

$$1 \quad vs. \quad 1 + \sqrt{\frac{K}{K+1}}$$

$$0 \quad vs. \quad \sqrt{\frac{k}{k+1}}$$

The right side is positive since k is positive, so the relationship is "<".

Since P_2 is true and P_K implies P_{K+1}, P_n is true for all $n \geq 2$.

P_n is $\dfrac{1\cdot 3\cdot 5\cdot \cdots (2n-1)}{2\cdot 4\cdot 6\cdot \cdots (2n)} \leq \dfrac{1}{\sqrt{n+1}}$ $\quad n \geq 1$

P_1 says that $\dfrac{1}{2} \leq \dfrac{1}{\sqrt{2}}$

which is equivalent to $2 \geq \sqrt{2}$ which is true.

Assume that P_k is true. In other words, assume that:

(1) $$\frac{1\cdot 3\cdot 5\cdot \cdots \cdot(2k-1)}{2\cdot 4\cdot 6\cdot \cdots \cdot (2k)} \leq \frac{1}{\sqrt{k+1}}.$$

for some $k \geq 1$.

Since $k \geq 1$, then clearly

$$0 \leq 3k+2.$$

$$\begin{array}{rcl} +\ 9k+2 & & +9k+2 \\ \hline 9k+2 & \leq & 12k+4 \\ +\ 4k^3+12k^2 & & +4k^3+12k^2 \\ \hline 4k^3+12k^2+9k+2 & \leq & 4k^3+12k^2+12k+4 \end{array}$$

And we can factor each side:

$$(4k^2+4k+1)(k+2) \leq (4k^2+8k+4)(k+1)$$

Divide both sides by $(4k^2+8k+4)(k+2)$:

$$\frac{4k^2+4k+1}{4k^2+8k+4} \leq \frac{k+1}{k+2}$$

Take square root of both sides.

(2) $$\frac{2k+1}{2k+2} \leq \sqrt{\frac{k+1}{k+2}}$$

Scratch work, not part of proof. P_{k+1} should look like this:

$$\frac{1\cdot 3\cdot 5\cdot \cdots (2k-1)(2k+1)}{2\cdot 4\cdot 6\cdots (2k)(2k+2)} \leq \frac{1}{\sqrt{k+2}}.$$

How does this relate to P_k?

Note that

$\dfrac{f(k+1)}{f(k)}$ $\underline{vs.}$ $\dfrac{g(k+1)}{g(k)}$ in this case is

whatever we just multiplied both sides of P_k to get P_{k+1}, which is

$\dfrac{2k+1}{2k+2}$ $\underline{vs.}$ $\sqrt{\dfrac{k+1}{k+2}}$ (for $k \geq 1$)

Since everything is positive, we can square both sides and preserve whatever the inequality is:

$\dfrac{4k^2+4k+1}{4k^2+8k+4}$ $\underline{vs.}$ $\dfrac{k+1}{k+2}$

Cross-multiply

$4k^3+12k^2+9k+2$ $\underline{vs.}$ $4k^3+12k^2+12k+4$

$9k+2$ $\underline{vs.}$ $12k+4$

0 $\underline{vs.}$ $3k+2$

The relationship is $\leq$ which is what we want.

Now, let's combine inequalities (1) and (2) by virtue of the basic rule "if $a \leq b$ and $c \leq d$ then $ac \leq bd$" and we obtain:

$$\frac{1\cdot 3\cdot 5\cdot \cdots \cdot(2k-1)}{2\cdot 4\cdot 6\cdot \cdots \cdot(2k)} \cdot \frac{2k+1}{2k+2} \leq \frac{1}{\sqrt{k+1}} \cdot \frac{\sqrt{k+1}}{\sqrt{k+2}}$$

which is equivalent to:

$$\frac{1\cdot 3\cdot 5\cdot \cdots \cdot(2k+1)}{2\cdot 4\cdot 6\cdot \cdots \cdot(2k+2)} \leq \frac{1}{\sqrt{k+2}}$$

And this is the statement P_{k+1}.

Thus, we have shown that P_1 is true and P_k implies P_{k+1}. Therefore P_n is true for all $n \geq 1$.

P_n says that $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\dots+\frac{n}{n+1} < \frac{n^2}{n+1}$ $n \geq 2$.

P_2 is the statement $\frac{1}{2}+\frac{2}{3} < \frac{2^2}{2+1}$

$\frac{7}{6} < \frac{4}{3}$ which is true.

Suppose P_k is true for some $k \geq 2$. That is, assume that

① $\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\dots+\frac{k}{k+1} < \frac{k^2}{k+1}$, for $k \geq 2$.

Since $k \geq 2$, it is clear that

$$0 < k$$

$$\underline{+k^2+2k+1 \qquad +k^2+2k+1}$$

$$k^2+2k+1 < k^2+3k+1$$

And we can divide both sides by $(k+1)(k+2)$:

$$\frac{(k+1)(k+1)}{(k+1)(k+2)} < \frac{k^2+3k+1}{(k+1)(k+2)}$$

$$\frac{k+1}{k+2} < \frac{k^2+3k+1}{(k+1)(k+2)}$$

d "zero":

$$\frac{k+1}{k+2} < \frac{k^3+3k^2+3k+1-k^3-2k^2}{(k+1)(k+2)}$$

Factor:

$$\frac{k+1}{k+2} < \frac{(k+1)^3-k^2(k+2)}{(k+1)(k+2)}$$

Resolve into partial fractions:

② $$\frac{k+1}{k+2} < \frac{(k+1)^2}{k+2} - \frac{k^2}{k+1}$$

We can now combine inequalities ① and ② by virtue of the basic rule "if $a<b$ and $c<d$, then $a+c<b+d$" to obtain:

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\dots+\frac{k}{k+1}+\frac{k+1}{k+2} < \frac{k^2}{k+1}+\frac{(k+1)^2}{k+2}-\frac{k^2}{k+1}$$

which is equivalent to:

$$\frac{1}{2}+\frac{2}{3}+\frac{3}{4}+\dots+\frac{k+1}{k+2} < \frac{(k+1)^2}{k+2}$$ and this is P_{k+1}.

Since P_2 is true and P_k implies P_{k+1}, then P_n is true for all $n \geq 2$.

Scratch work, not part of the proof.

P_k is a statement of the form $f(k) < g(k)$. So P_{k+1} looks like $f(k+1) < g(k+1)$ which would be

$$\frac{1}{2}+\frac{2}{3}+\dots\frac{k+1}{k+2} < \frac{(k+1)^2}{k+2}.$$

Consider a comparison between $f(k+1)-f(k)$ and $g(k+1)-g(k)$.

We would like to use the fact that "if $a<b$ and $c<d$, then $a+c<b+d$"

$$f(k+1)-f(k) \text{ vs. } g(k+1)-g(k)$$

$$\frac{k+1}{k+2} \text{ vs. } \underbrace{\frac{(k+1)^2}{k+2}-\frac{k^2}{k+1}}_{\frac{(k+1)^3-k^2(k+2)}{(k+1)(k+2)}}$$

So the right side simplifies to

$$\frac{k^3+3k^2+3k+1-k^3-2k^2}{(k+1)(k+2)} = \frac{k^2+3k+1}{(k+1)(k+2)}$$

$$\frac{k+1}{k+2} \text{ vs. } \frac{k^2+3k+1}{(k+1)(k+2)}$$

$$\frac{k+1}{1} \text{ vs. } \frac{k^2+3k+1}{k+1}$$

cross-multiply

$$k^2+2k+1 \text{ vs. } k^2+3k+1$$

$$0 \text{ vs. } k$$

The relationship is $<$ since $k \geq 2$.