

Here is an example of a nonhomogeneous recurrence relation where the particular solution is a quadratic function. Let's solve it from scratch.

$$a_n = 5a_{n-1} - 6a_{n-2} + 2n^2 - 26, \text{ with base cases } a_0 = 5 \text{ and } a_1 = 20$$

When we bring the sequence terms to the left side of the equation, we have

$a_n - 5a_{n-1} + 6a_{n-2} = 2n^2 - 26$ , and we see that the right side is a 2<sup>nd</sup> degree polynomial. Therefore, the form of our particular solution also needs to be some 2<sup>nd</sup> degree polynomial. So, we let our particular solution have the form  $an^2 + bn + c$ . This is  $a_n$ . We need to also determine analogous expressions for  $a_{n-1}$  and  $a_{n-2}$ . All we need to do is substitute  $n - 1$  and  $n - 2$  for  $n$ :

$$a_n = an^2 + bn + c, \quad a_{n-1} = a(n-1)^2 + b(n-1) + c, \quad a_{n-2} = a(n-2)^2 + b(n-2) + c$$

We are ready to substitute into the recursive rule. This will allow us to solve for the coefficients  $a$ ,  $b$  and  $c$  and thereby determine the particular solution of the recurrence. On the left side of the equation, we need to take great care to collect like terms so that we can match up the coefficients of  $n^2$ ,  $n$  and the constant terms on both sides of the equation.

$$a_n - 5a_{n-1} + 6a_{n-2} = 2n^2 - 26$$

$$(an^2 + bn + c) - 5(a(n-1)^2 + b(n-1) + c) + 6(a(n-2)^2 + b(n-2) + c) = 2n^2 - 26$$

$$(an^2 + bn + c) - 5(an^2 - 2an + a + bn - b + c) + 6(an^2 - 4an + 4a + bn - 2b + c) = 2n^2 + 0n - 26$$

$$(a - 5a + 6a)n^2 + (b + 10a - 5b - 24a + 6b)n + (c - 5a + 5b - 5c + 24a - 12b + 6c) = 2n^2 + 0n - 26$$

$$(2a)n^2 + (-14a + 2b)n + (19a - 7b + 2c) = 2n^2 + 0n - 26$$

We want this equation to be true for all  $n$ . Since corresponding coefficients must match, we obtain this system of equations:

$$2a = 2$$

$$-14a + 2b = 0$$

$$19a - 7b + 2c = -26$$

The first equation tells us that  $a = 1$ . Substituting into the second equation, we have  $-14 + 2b = 0$ , which works out to  $2b = 14$  or  $b = 7$ . Finally, substituting into the final equation we have  $19 - 49 + 2c = -26$ , which simplifies to  $2c = 4$  or  $c = 2$ . Therefore,  $a = 1$ ,  $b = 7$  and  $c = 2$ , so that our particular solution is  $a_n = n^2 + 7n + 2$ .

For the homogeneous solution, we solve the quadratic equation  $x^2 - 5x + 6 = 0$ . This can be factored as  $(x - 2)(x - 3) = 0$ , so our solutions are  $x = 2$  and  $x = 3$ . We are now ready to write the general form of our entire solution, which is  $a_n = c_1 2^n + c_2 3^n + n^2 + 7n + 2$ . The base cases will help us determine  $c_1$  and  $c_2$ .

If  $n = 0$ , we have  $c_1 2^0 + c_2 3^0 + 0^2 + 7(0) + 2 = 5$ , which simplifies to  $c_1 + c_2 = 3$ .

If  $n = 1$ , we have  $c_1 2^1 + c_2 3^1 + 1^2 + 7(1) + 2 = 20$ , which simplifies to  $2c_1 + 3c_2 = 10$ .

If we multiply the first equation by  $-2$  and add it to the second, we will obtain  $c_2 = 4$ . And substituting into the first equation we see that  $c_1 = -1$ . Therefore, our final answer is

$$a_n = (-1) 2^n + (4) 3^n + n^2 + 7n + 2$$