$\forall n \in \mathbb{Z},\ n$ is odd $\longrightarrow n^2$ is odd

Let n be an odd integer. Then,

$n = 2k+1$ for some integer k.

Consider the number n^2.

$$n^2 = (2k+1)^2 = 4k^2+4k+1$$
$$= 2\underbrace{(2k^2+2k)} + 1$$

Notice that $2k^2+2k$ is an integer.

Therefore n^2 is odd by definition, because it can be written in the form $n^2 = 2a+1$ where $a = 2k^2+2k$.

The product of any 2 consec. int is even.

Let the integers be n and $n+1$.

Their product is $n(n+1)$. $= n^2+n$

There are 2 cases.

Suppose n is odd. Then

$n = 2k+1$ for some int k. and

$n+1 = 2k+2$

Then $n(n+1) = (2k+1)(2k+2) = 4k^2+6k+2$

$$= 2(2k^2+3k+1)$$

Since $(2k^2+3k+1)$ is int, $n(n+1)$ satisfies the defn of even.

2nd case. Suppose n is even. Then,

$n = 2k$ for some int k.

$n+1 = 2k+1$

Therefore, $n(n+1) = 2k(2k+1) = 2\big(k(2k+1)\big)$

which is clearly an even number because it is 2 mult by an integer $k(2k+1)$.

We see in both cases $n(n+1)$ is always even.

For all odd int n, $5n+3$ is even.

Since n is odd, $n = 2k+1$ for some integer k.

Consider the number $5n+3$.

$$
\begin{aligned}
5n+3 &= 5(2k+1)+3 \\
&= 10k+5+3 \\
&= 10k+8 = 2(5k+4).
\end{aligned}
$$

Since $5k+4$ is an int, we see that $5n+3$ satisfies the definition of even.

For any int n, $3 \mid$ sum of $n, n+1, n+2$

Let n be an integer. Consider the sum

$$
\begin{aligned}
n+(n+1)+(n+2) &= 3n+3 \\
&= 3(n+1).
\end{aligned}
$$

This number is clearly divisible by 3 because $n+1$ is an integer.

Let a, b, c be integers.
Need to show if $\underbrace{a|b \text{ and } b|c}$ then $a|c$.

Since $a|b$, then there is some int k : $\boxed{ak = b}$
$b|c$, " . . . . r : $\boxed{br = c}$

Consider the number c.

$c = br$ and we notice $b = ak$, so that we can subst.

$c = (ak)r = akr$.

Is c an integer ~~to~~ multiple of a?
Yes because kr is an integer
Therefore, by defn, $a|c$.

If $a|b$ and $a|c$, then $a|(b+c)$

Let a, b, c be int such that $a|b$ and $a|c$.
Since $a|b$, $\exists$ some int k for which $ak = b$
$a|c$, ... r . . . . $ar = c$

Consider the number $b+c$

$b+c = (ak) + (ar)$ substituting from above

$= a(k+r)$.

Is $b+c$ an int multiple of a?
Yes, that multiple is $k+r$.
So, by defn, $a|(b+c)$.

For all int a, b, c, if $a | bc$, then $a|b$ or $a|c$.

~~Let~~ Since $a | bc$, $\exists$ int k s.t. $ak = bc$

we need to show $b = a\,(\text{integer})$

or $c = a\,(\text{integer})$.

$$ak = bc \implies b = \frac{ak}{c}$$

Let $a = 4$, $b = 2$ and $c = 2$.

Then $a | bc$ but a divides neither b nor c.

So, the statement is disproved.

HW 6 is 4.2 # 5, 8, 27

<u>Review definitions</u>

An integer n is odd if $\exists$ int k s.t. $n = 2k+1$

An integer n is even if $\exists$ int k s.t. $n = 2k$

~~If d and n are integers,~~

~~we say that d|n if~~

If d and n are integers,

we say that $d | n$ if $\exists$ int k s.t. $n = dk$.

In programming, we'd say $n \% 2 == 1$ $\leftarrow$ odd

$n \% 2 == 0$ $\leftarrow$ even

$n \% d == 0$ $\leftarrow$ $d | n$

~~The number~~

~~A number x is rational if $x = \frac{a}{b}$ where~~

$\forall$ int x, y, if x and y odd, then $6 \mid (3x+3y)$.

Since x, y odd, $\exists$ int m and n s.t.

$x = 2m+1$

$y = 2n+1$

Consider the number $3x+3y$

$$3x + 3y = 3(2m+1) + 3(2n+1)$$
$$= 6m + 3 + 6n + 3 = 6m + 6n + 6$$
$$= 6(m+n+1)$$

Note that $m+n+1$ is int. Therefore, by defn, $3x+3y$ is divis by 6

For any 3 consec odd int, 3 divides one of them.

Let $n, n+2, n+4$ be three consec odd int.

Suppose $n \bmod 3 = 0$ ←
then, $(n+2) \bmod 3 = 2$
$(n+4) \bmod 3 = 1$

Suppose $n \bmod 3 = 1$
then $(n+2) \bmod 3 = 0$ ←
$(n+4) \bmod 3 = 2$

Suppose $n \bmod 3 = 2$.
then $(n+2) \bmod 3 = 1$
$(n+4) \bmod 3 = 0$ ←

In each case, we notice that there is always a number divis by 3.

which is what we were asked to show.

Every even int is $\frac{2}{3}$ of some other integer.

i.e. For all even int n, $\exists$ some int k s.t. $n = \frac{2}{3}k$.

Let n be even. Then $n = 2a$ for some int a.

Consider the number k s.t. $n = \frac{2}{3}k$.

$$k = \frac{3}{2}n$$
$$= \frac{3}{2}(2a) = 3a.$$

Therefore, k is an int because it's the product of 2 ints, 3 and a.

If $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \dots = \frac{a_n}{b_n} = r$

show that $\frac{a_1 + a_2 + \dots + a_n}{b_1 + b_2 + \dots + b_n} = r$

Get rid of fractions.

We have
$$\left.\begin{aligned} a_1 &= r\,b_1 \\ a_2 &= r\,b_2 \\ a_3 &= r\,b_3 \\ &\vdots \\ a_n &= r\,b_n \end{aligned}\right\} \text{ for subst below}$$

Consider the big numerator only.

$$a_1 + a_2 + a_3 + \dots + a_n = rb_1 + rb_2 + rb_3 + \dots + rb_n$$
$$= r(b_1 + b_2 + b_3 + \dots + b_n)$$

Therefore, $\frac{a_1 + a_2 + \dots + a_n}{b_1 + b_2 + \dots + b_n} = \frac{r(b_1 + b_2 + \dots + b_n)}{b_1 + b_2 + \dots + b_n} = r$

joke Simplify $(x-a)(x-b)(x-c)\dots(x-z) = 0$

Prove: $\nexists$ an integer both odd & even.

~~Pf~~ We prove by contradiction. Suppose the given stmt is false. Then, there is some int n that is odd and even.

Since n is odd, $n = 2k+1$ for some int k.
Since n is even, $n = 2u$ for some int u.

Then, $2k+1 = 2u$

$$2k - 2u = -1$$

$$2u - 2k = 1$$

BTW, we'd conclude 1 is even. $\leftarrow$ $2(u-k) = 1$

$$u - k = 1/2$$

When we sub int, we should get int, but $1/2$ is not int. Contradiction.

Therefore the orig stmt must be true.

There does not exist a positive real # x s.t. $3x^5 + 7x + 6 = 0$.

~~Pf~~ i.e. Suppose the given stmt does not hold. Then, there is some positive real x s.t.

$$\underbrace{3x^5 + 7x + 6}_{>0} = \underbrace{0}_{=0}.$$

since $x > 0$,

All these terms are > 0.

Contradiction.

Therefore, orig stmt is true.

$$\sum_{i=1}^{4}(2i+3) = \underline{2\cdot1+3} + \underline{2\cdot2+3} + \underline{2\cdot3+3} + \underline{2\cdot4+3}$$

too slow $\cancel{= 5 + 8 + 11 +}$

$$= 5 + 7 + 9 + 11 \qquad = 32$$

$$\boxed{\sum_{i=1}^{n} 1 = n} \qquad \boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}}$$

$$\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}} \qquad \boxed{\sum_{i=1}^{n} i^3 = \frac{n^2(n+1)^2}{4}}$$

What is the sum of the first 17 perfect squares?

$$1^2 + 2^2 + \dots + 17^2 = \frac{17(17+1)(2\cdot17+1)}{6}$$

$$= 17\cdot18\cdot35/6 = 51\cdot35 = 1785$$

A health club owns a set of dumbbells. They are pairs of dumbbells in increments of 5 lbs. from 5 thru 100 lbs. What is their total weight?

$$(5+5) + (10+10) + \dots + (100+100)$$

$$= 2(5 + 10 + 15 + \dots + 100)$$

$$= 2\cdot5(1 + 2 + 3 + \dots + 20)$$

$$= 10 \sum_{i=1}^{20} i = 10\cdot\frac{20(21)}{2} = 2100$$

First $\overset{n}{100}$ terms of $10i^2 + 9i + 2$

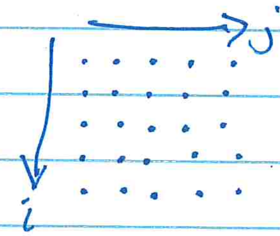
$$10\sum i^2 + 9\sum i + 2\sum 1$$

$$10\left[\frac{100(101)(201)}{6}\right] + 9\left[\frac{100(101)}{2}\right] + 2\left[100\right]$$

	total operations
$i = 1$	1
while $(i <= n)$:	$n+1$
1	n
2	n
3 stmts $i = i+1$	n
	$4n+2$

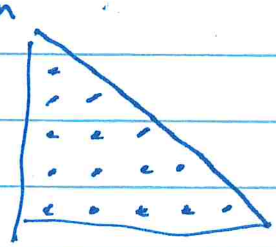
	total ops
$i = 1$	1
while $i <= n$:	$n+1$
$j = 1$	n
while $j <= n$:	$n(n+1)$
①	n^2
②	n^2
③ stmts $j = j+1$	n^2
	$\underline{4n^2} + 3n + 2$

inner loop



$i = 1$	1
while $i <= n$:	$n+1$
$j = 1$	n
while $j <= i$:	$n + \Sigma i = \Sigma(i+1)$
①	$1+2+3+\dots n = \Sigma i$
②	$1+2+3+\dots n \quad \Sigma i$
③ $j = j+1$	$1+2+3+\dots n = \Sigma i$

enter inner loop n times
$i=1$
$i=2$
$\vdots$
$i=n$



$$4\Sigma i + 3n + 2$$

$$4\left[\frac{n(n+1)}{2}\right] + 3n + 2$$

$$2n(n+1) + 3n + 2$$

$$\underline{2n^2} + 5n + 2$$

Prove

The sum of the first n odd pos int is n^2.

P_n: $\forall\, n \geq 1, \quad 1 + 3 + 5 + \cdots + (2n-1) = n^2$

P_1 says $1 = 1^2$, which is true.

Next, assume that P_k is true. That is, assume that for some arbitrary $k \geq 1$,

$$1 + 3 + 5 + \cdots + (2k-1) = k^2$$

☆ Let's add the next term of the series to both sides $+ (2k+1) \quad + (2k+1)$

"$k+1$" term is $2(k+1) - 1$
$= 2k+1$

We obtain $1 + 3 + 5 + \cdots + (2k-1) + (2k+1) = k^2 + 2k + 1$
$= (k+1)^2$

This equation is P_{k+1}.

Since P_1 is true and $P_k \Rightarrow P_{k+1}$, P_n is true $\forall\, n \geq 1$.

P_n: $\forall\, n \geq 1, \quad 1^2 + 2^2 + 3^2 + \ldots + n^2 = \dfrac{n(n+1)(2n+1)}{6}$

First we show the base case is true:

P_1 says $\quad 1^2 = \dfrac{1(1+1)(2+1)}{6}$

$1^2 = \dfrac{6}{6}$ true!

Next, assume P_k is true for some arbitrary $k \geq 1$.

i.e. assume

$$1^2 + 2^2 + 3^2 + \ldots + k^2 = \frac{k(k+1)(2k+1)}{6}$$

we add the ~~#~~ term ~~#~~ $(k+1)$ to both sides, $(k+1)^2$ $\quad +(k+1)^2 \qquad + (k+1)^2$

We obtain

$$1^2 + 2^2 + \ldots + k^2 + (k+1)^2 = \frac{k(k+1)(2k+1)}{6} + (k+1)^2$$

$$= \frac{k+1}{6}\left[k(2k+1) + 6(k+1)\right]$$

$$= \frac{k+1}{6}\left(2k^2 + 7k + 6\right)$$

$$= \frac{k+1}{6}(k+2)(2k+3)$$

$$= \frac{(k+1)(k+2)(2k+3)}{6}$$

This last eqn is P_{k+1}.

Since P_1 is true and $P_k \Rightarrow P_{k+1}$, then P_n is true $\forall\, n \geq 1$.

$\forall n \geq 1, \quad 1(1!) + 2(2!) + 3(3!) + \ldots + n(n!) = (n+1)! - 1$

P_1 says $\quad 1(1!) = 2! - 1$

$\qquad\qquad 1 = 2 - 1$ ✓

Assume P_K is true for some arbitrary $k \geq 1$.

$$1(1!) + 2(2!) + \ldots k(k!) = \underline{(k+1)! - 1}$$

Add next term to both sides

$$+ (k+1)(k+1)! \qquad \underline{(k+1)(k+1)!}$$

We obtain

$$1(1!) + 2(2!) + \ldots + (k+1)(k+1)! =$$

$$= (k+1)! - 1 + (k+1)(k+1)!$$

$$= (k+1)! + (k+1)(k+1)! - 1$$

$$= (k+1)! [1 + k+1] - 1$$

$$= (k+2)(k+1)! - 1$$

$$= (k+2)! - 1$$

This last eqn is P_{K+1}.

Since P_1 true and $P_K \rightarrow P_{K+1}$, P_n true $\forall n \geq 1$.

$\forall\, n \geq 14,\ \forall\, x, y$ s.t. $n = 3x + 8y$

Base case. $14 = 3(2) + 8(1)$

Assume $k = 3x + 8y$ for some $k \geq 14$.

Observe that

$$15 = 3(5) + 8(0)$$
$$16 = 3(0) + 8(2)$$

gain 3 loss 1

~~loss~~ loss 5 gain 2

if $y \geq 1$

$k+1 = 3(x+3) + 8(y-1)$

else

$k+1 = 3(x-5) + 8(y+2)$

$\forall\, n \geq 0,\ 6 \mid n^3 - n.$

Base case. Let $n = 0$. Then, we have $6 \mid (0^3 - 0)$

$6 \mid 0$ ✓

Assume that for an arbitrary $k \geq 0$, $6 \mid k^3 - k$. ☆

We need to prove $6 \mid (k+1)^3 - (k+1)$.

Consider the difference

$$[(k+1)^3 - (k+1)] - [k^3 - k] \quad ☆$$
$$= \cancel{k^3} + 3k^2 + 3k + 1 - k - 1 - \cancel{k^3} + k$$
$$= 3k^2 + 3k$$
$$= 3\underbrace{k(k+1)}_{\text{one of these factors is even.}}.$$

$\Rightarrow$ clearly a mult of 6

Since $6 \mid k^3 - k$ and

$6 \mid [(k+1)^3 - (k+1)] - [k^3 - k]$,

6 divides their sum, which is

$$6 \mid (k+1)^3 - (k+1)$$

This is $P(k+1)$.

Since $P(0)$ is true and $P(k) \Rightarrow P(k+1)$,

$P(n)$ is true $\forall\ n \geq 0$.